Riemann surfaces

lecture 6: hyperbolic plane

Misha Verbitsky

Université Libre de Bruxelles

November 10, 2015

Riemannian manifolds (reminder)

DEFINITION: Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: For any $x.y \in M$, and any path $\gamma: [a,b] \longrightarrow M$ connecting x and y, consider **the length** of γ defined as $L(\gamma) = \int_\gamma |\frac{d\gamma}{dt}|dt$, where $|\frac{d\gamma}{dt}| = h(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2}$. Define **the geodesic distance** as $d(x,y) = \inf_\gamma L(\gamma)$, where infimum is taken for all paths connecting x and y.

EXERCISE: Prove that the **geodesic distance satisfies triangle inequality and defines metric on M.**

EXERCISE: Prove that **this metric induces the standard topology on M.**

EXAMPLE: Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x,y) = |x-y|$.**

EXERCISE: Using partition of unity, **prove that any manifold admits a Riemannian structure.**

Hermitian structures (reminder)

DEFINITION: A Riemannia metric h on an almost complex manifold is called **Hermitian** if $h(x, y) = h(Ix, Iy)$.

REMARK: Given any Riemannian metric g on an almost complex manifold, **a Hermitian metric h can be obtained as $h = g + I(g)$, where $I(g)(x, y) = g(I(x), I(y))$.**

REMARK: Let I be a complex structure operator on a real vector space V, and g – a Hermitian metric. Then **the bilinear form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.** Indeed, $\omega(x, y) = g(x, Iy) = g(Ix, I^2y) = -g(Ix, y) = -\omega(y, x)$.

DEFINITION: A skew-symmetric form $\omega(x, y)$ is called **an Hermitian form on (V, I)**.

REMARK: In the triple I, g, ω, each element can recovered from the other two.

Conformal structure (reminder)

DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

DEFINITION: **Conformal structure** on M is a class of conformal equivalence of Riemannian metrics.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**. Conversely, any metric conformally equivalent to Hermitian is Hermitian.

REMARK: The last statement is clear from the definition, and true in any dimension.

Conformal structures and almost complex structures (reminder)

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I, let ν be the conformal class of its Hermitian metric. **Then ν is determined by I, and it determines I uniquely.**

DEFINITION: A Riemann surface is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure. A map from one Riemann surface to another is holomorphic if and only if it preserves the conformal structure.

Homogeneous spaces (reminder)

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\operatorname{St}_x(G) = \{g \in G \ \mid \ g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \operatorname{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\operatorname{St}_x(G)^g = \operatorname{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation (reminder)

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\mathrm{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\mathrm{St}_x(G)$ on T_xM.

DEFINITION: A tensor Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant tensor on $\mathrm{St}_x(G)$. For any $y \in M$ obtained as $y = g(x)$, consider the tensor Φ_y on T_yM obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \mathrm{St}_x(G)$. Since Φ is h-invariant, **the tensor Φ_y is independent from the choice of g.**

We proved

THEOREM: Homogeneous tensors on $M = G/H$ are in bijective correspondence with isotropy invariant tensors on T_xM, for any $x \in M$.

■

Space forms (reminder)

DEFINITION: Simply connected space form is a homogeneous manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

Riemannian metric on space forms (reminder)

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique G-invariant symmetric 2-form:** the standard Euclidean metric.

Proof: Let g, g' be two G-invariant symmetric 2-forms. Since S^{n-1} is an orbit of G, we have $g(x,x) = g(y,y)$ for any $x, y \in S^{n-1}$. Multiplying g' by a constant, we may assume that $g(x,x) = g'(x,x)$ for any $x \in S^{n-1}$. **Then $g(\lambda x, \lambda x) = g'(\lambda x, \lambda x)$ for any $x \in S^{n-1}$, $\lambda \in \mathbb{R}$;** however, all vectors can be written as λx. ■

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■

REMARK: From now on, all space forms are assumed to be homogeneous Riemannian manifolds.

Some low-dimensional Lie group isomorphisms (reminder)

DEFINITION: Lie algebra of a Lie group G is the Lie algebra Lie(G) of left-invariant vector fields. **Adjoint representation** of G is the standard action of G on Lie(G). For a Lie group $G = GL(n)$, $SL(n)$, etc., $PGL(n)$, $PSL(n)$, etc. denote the image of G in $GL($Lie$(G))$ with respect to the adjoint action.

REMARK: This is the same as a quotient G/Z by the centre of G.

DEFINITION: Define $SO(1,2)$ as the group of orthogonal matrices on a 3-dimensional space equipped with a scalar product of signature (1,2), and $U(1,1)$ as the group of complex linear maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ preserving a pseudio-Hermitian form of signature (1,1).

THEOREM: The groups $PU(1,1)$, $PSL(2,\mathbb{R})$ and $SO(1,2)$ are isomorphic.

Proof: Isomorphism $PU(1,1) = SO(1,2)$ will be established later in this lecture. To see $PSL(2,\mathbb{R}) \cong SO(1,2)$, consider **the Killing form** κ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, $a,b \longrightarrow \operatorname{Tr}(ab)$. **Check that it has signature $(1,2)$. Then the image of $SL(2,\mathbb{R})$ in automorphisms of its Lie algebra is mapped to $SO(\mathfrak{sl}(2,\mathbb{R}),\kappa) = SO(1,2)$.** Both groups are 3-dimensional, hence it is an isomorphism. ■

Poincaré-Koebe uniformization theorem (reminder)

DEFINITION: A **Riemannian manifold of constant curvature** is a Riemannian manifold which is locally isometric to a space form.

THEOREM: (Poincaré-Koebe uniformization theorem) Let M be a Riemann surface. **Then M admits a unique complete metric of constant curvature in the same conformal class.**

COROLLARY: Any Riemann surface is a quotient of a space form X by a discrete group of isometries $\Gamma \subset \operatorname{Iso}(X)$.

COROLLARY: Any simply connected Riemann surface is conformally equivalent to a space form.

REMARK: We shall prove some cases of the uniformization theorem in later lectures.

Today's subject: classify conformal automorphisms of all space forms.

Laurent power series

THEOREM: (Laurent theorem)
Let f be a holomorphic function on an annulus (that is, a ring)

$$R = \{z \mid \alpha < |z| < \beta\}.$$

Then f can be expressed as a Laurent power series $f(z) = \sum_{i\in\mathbb{Z}} z^i a_i$ converging in R.

Proof: Same as Cauchy formula. ■

REMARK: This theorem remains valid if $\alpha = 0$ and $\beta = \infty$.

REMARK: A function $\varphi: \mathbb{C}^* \longrightarrow \mathbb{C}$ uniquely determines its Laurent power series. Indeed, residue of $z^k\varphi$ in 0 is $\sqrt{-1}\, 2\pi a_{-k-1}$.

REMARK: Let $\varphi: \mathbb{C}^* \longrightarrow \mathbb{C}$ be a holomorphic function, and $\varphi = \sum_{i\in\mathbb{Z}} z^i a_i$ its Laurent power series. **Then $\psi(z) := \varphi(z^{-1})$ has Laurent polynomial $\psi = \sum_{i\in\mathbb{Z}} z^{-i} a_i$.**

Affine coordinates on $\mathbb{C}P^1$

DEFINITION: We identify $\mathbb{C}P^1$ with the set of pairs $x : y$ defined up to equivalence $x : y \sim \lambda x : \lambda y$, for each $\lambda \in \mathbb{C}^*$. This representation is called **homogeneous coordinates**. **Affine coordinates** are $1 : z$ for $x \neq 0$, $z = y/x$ and $z : 1$ for $y \neq 0$, $z = x/y$. The corresponding gluing functions are given by the map $z \longrightarrow z^{-1}$.

DEFINITION: **Meromorphic function** is a quotient f/g, where f, g are holomorphic and $g \neq 0$.

REMARK: A holomorphic map $\mathbb{C} \longrightarrow \mathbb{C}P^1$ is the same as a pair of maps $f : g$ up to equivalence $f : g \sim fh : gh$. **In other words, holomorphic maps $\mathbb{C} \longrightarrow \mathbb{C}P^1$ are identified with meromorphic functions on $\mathbb{C}$.**

REMARK: In homogeneous coordinates, an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$ acts as $x : y \longrightarrow ax + by : cx + dy$. Therefore, in affine coordinates it acts as $z \longrightarrow \frac{az+b}{cz+d}$.

Möbius transforms

DEFINITION: Möbius transform is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: The group $PGL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphially.

The following theorem will be proven later in this lecture.

THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group of Möbius transforms is an isomorphism.

Claim 1: Let $\varphi: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a holomorphic automorphism, $\varphi_0 : \mathbb{C} \longrightarrow \mathbb{C}P^1$ its restriction to the chart $z : 1$, and $\varphi_\infty : \mathbb{C} \longrightarrow \mathbb{C}P^1$ its restriction $1 : z$. We consider φ_0, φ_∞ as meromorphic functions on $\mathbb{C}$. **Then $\varphi_\infty = \varphi_0(z^{-1})^{-1}$.**

Möbius transforms and $PGL(2,\mathbb{C})$

THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group $\operatorname{Aut}(\mathbb{C}P^1)$ of Möbius transforms is an isomorphism.

Proof. Step 1: Let $\varphi \in \operatorname{Aut}(\mathbb{C}P^1)$. Since $PSL(2,\mathbb{C})$ acts transitively on pairs of points $x \neq y$ in $\mathbb{C}P^1$, by composing φ with an appropriate element in $PGL(2,\mathbb{C})$ we can assume that $\varphi(0) = 0$ and $\varphi(\infty = \infty$. This means that we may consider the restrictions φ_0 and φ_∞ of φ to the affine charts as a holomorphic functions on these charts, $\varphi_0, \varphi_\infty : \mathbb{C} \longrightarrow \mathbb{C}$.

Step 2: Let $\varphi_0 = \sum_{i>0} a_i z^i$, $a_1 \neq 0$. Claim 1 gives

$$\varphi_\infty(z) = \varphi_0(z^{-1})^{-1} = a_1 z(1 + \sum_{i\geqslant 2} \frac{a_i}{a_1} z^{-i})^{-1}.$$

Unless $a_i = 0$ for all $i \geqslant 2$, this Laurent series has singularities in 0 and cannot be holomorphic. **Therefore φ_0 is a linear function**, and it belongs to $PGL(2,\mathbb{C})$. ■

Lemma 1: Let φ be a Möbius transform fixing $\infty \in \mathbb{C}P^1$. **Then $\varphi(z) = az + b$ for some $a, b \in \mathbb{C}$ and all $z = z : 1 \in \mathbb{C}P^1$.**
Proof: Let $A \in PGL(2,\mathbb{C})$ be a map acting on $\mathbb{C} = \mathbb{C}P^1 \backslash \infty$ as parallel transport mapping $\varphi(0)$ to 0. Then $\varphi \circ A$ is a Moebius transform which fixes ∞ and 0. As shown in Step 2 above, it is a linear function. ■

Properties of Möbius transform

DEFINITION: A circle in S^2 is an orbit of a 1-parametric isometric rotation subgroup $U \subset PGL(2,\mathbb{C})$.

PROPOSITION: The action of $PGL(2,\mathbb{C})$ on $\mathbb{C}P^1$ maps circles to circles.

Proof. Step 1: Consider a pseudo-Hermitian form h on $V = \mathbb{C}^2$ of signature (1,1). Let h_+ be a positive definite Hermitian form on V. There exists a basis $x, y \in V$ such that $h_+ = \sqrt{-1}\, x \otimes \overline{x} + \sqrt{-1}\, y \otimes \overline{y}$ (that is, x, y is orthonormal with respect to h_+) and $h = -\sqrt{-1}\,\alpha x \otimes \overline{x} + \sqrt{-1}\,\beta y \otimes \overline{y}$, with $\alpha > 0$, $\beta < 0$ real numbers. Then $\{z \mid h(z,z) = 0\}$ is invariant under the rotation $x, y \longrightarrow x, e^{\sqrt{-1}\,\theta} y$, hence **it is a circle.**

Step 2: Clearly, all circles are obtained this way.

Step 3: $PGL(2,\mathbb{C})$ maps pseudo-Hermitian forms to pseudo-Hermitian forms of the same signature, and therefore **preserves circles.** ■

Orbits of compact one-parametric subgroups in $PSL(2,\mathbb{C})$

LEMMA: Let $G \cong S^1$ be a compact one-parametric subgroup in $PSL(2,\mathbb{C})$. **Then any G-orbit in $\mathbb{C}P^1$ is a circle.**

Proof. Step 1: Let $V = \mathbb{C}^2$, and consider the natural projection map $\pi : SL(V) \longrightarrow PSL(2,\mathbb{C}) = SL(V)/\pm 1$. Then $\tilde{G} = \pi^{-1}(G)$ is compact. Choose a $\tilde{G}$-invariant Hermitian metric h_1 on V, and let h be the standard Hermitiann metric. Since $GL(2,\mathbb{C})$ acts on the set of Hermitian metrics transitively **(prove it)**, there exists $u \in GL(V)$ such that $u(h) = h_1$. By definition, circles on $\mathbb{C}P^1$ are orbits of one-parametric subgroups in $U(V,h)$. **Since $u(\tilde{G})$ is a one-parametric subgroup in $U(V,h)$, its orbit is a circle.**

Step 2: From Step 1, we obtain that any orbit of G is u^{-1}(*circle*). Since u^{-1} is a Moebius transform, and Moebius transforms preserve circles, this orbit is a circle. ■

Conformal automorphisms of $\mathbb{C}$

THEOREM: (Riemann removable singularity theorem) Let $f: \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function which is holomorphic outside of a finite set. **Then f is holomorphic.**

Proof: Use the Cauchy formula. ∎

THEOREM: All conformal automorphisms of $\mathbb{C}$ can be expressed by $z \longrightarrow az + b$, where a, b are complex numbers, $a \neq 0$.

Proof: Let φ be a conformal automorphism of $\mathbb{C}$. The Riemann removable singularity theorem implies that **φ can be extended to a holomorphic automorphism of $\mathbb{C}P^1$.** Indeed, $\mathbb{C}P^1$ is obtained as a 1-point compactification of $\mathbb{C}$, and any continuous map from $\mathbb{C}$ to $\mathbb{C}$ is extended to a continuous map on $\mathbb{C}P^1$. Now, Lemma 1 implies that $\varphi(z) = az + b$. ∎

Schwartz lemma

CLAIM: (maximum principle) Let f be a holomorphic function defined on an open set U. **Then f cannot have strict maxima in U. If f has non-strict maxima, it is constant.**

EXERCISE: Prove the maximum principle.

LEMMA: (Schwartz lemma) Let $f: \Delta \longrightarrow \Delta$ be a map from disk to itself fixing 0. **Then $|f'(0)| \leqslant 1$, and equality can be realized only if $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.**

Proof: Consider the function $\varphi := \frac{f(z)}{z}$. Since $f(0) = 0$, it is holomorphic, and since $f(\Delta) \subset \Delta$, on the boundary $\partial\Delta$ we have $|\varphi||_{\partial\Delta} \leqslant 1$. Now, **the maximum principle implies that $|f'(0)| = |\varphi(0)| \leqslant 1$,** and equality is realized only if $\varphi = const$. ■

Conformal automorphisms of the disk act transitively

CLAIM: Let $\Delta \subset \mathbb{C}$ be the unit disk. **Then the group $\operatorname{Aut}(\Delta)$ of its holomorphic automorphisms acts on Δ transitively.**

Proof. Step 1: Let $V_a(z) = \frac{z-a}{1-\overline{a}z}$ for some $a \in \Delta$. Then $V_a(0) = -a$. To prove transitivity, it remains to show that $V_a(\Delta) = \Delta$.

Step 2: For $|z| = 1$, we have

$$|V_a(z)| = |V_a(z)||z| = \left|\frac{z\overline{z} - a\overline{z}}{1 - \overline{a}z}\right| = \left|\frac{1 - a\overline{z}}{1 - \overline{a}z}\right| = 1.$$

Therefore, V_a preserves the circle. Maximum principle implies that V_a maps its interior to its interior.

Step 3: To prove invertibility, we interpret V_a as an element of $PGL(2, \mathbb{C})$. ■

Transitive action is determined by a stabilizer of a point

Lemma 2: Let $M = G/H$ be a homogeneous space, and $\Psi: G_1 \longrightarrow G$ a homomorphism such that G_1 acts on M transitively and $\mathrm{St}_x(G_1) = \mathrm{St}_x(G)$. **Then $G_1 = G$.**

Proof: Since any element in $\ker\Psi$ belongs to $\mathrm{St}_x(G_1) = \mathrm{St}_x(G) \subset G$, the homomorphism Ψ is injective. It remais only to show that Ψ is surjective.

Let $g \in G$. Since G_1 acts on M transitively, $gg_1(x) = x$ for some $g_1 \in G_1$. Then $gg_1 \in \mathrm{St}_x(G_1) = \mathrm{St}_x(G) \subset \operatorname{im} G_1$. This gives $g \in G_1$. ■

Group of conformal automorphisms of the disk

REMARK: The group $PU(1,1) \subset PGL(2,\mathbb{C})$ of unitary matrices preserving a pseudo-Hermitian form h of signature (1,1) acts on a disk $\{l \in \mathbb{C}P^1 \mid h(l,l) > 0\}$ by holomorphic automorphisms.

COROLLARY: Let $\Delta \subset \mathbb{C}$ be the unit disk, $\operatorname{Aut}(\Delta)$ the group of its conformal automorphisms, and $\Psi : PU(1,1) \longrightarrow \operatorname{Aut}(\Delta)$ the map constructed above. **Then Ψ is an isomorphism.**

Proof: We use Lemma 2. Both groups act on Δ transitively, hence **it suffices only to check that $\operatorname{St}_x(PU(1,1)) = S^1$ and $\operatorname{St}_x(\operatorname{Aut}(\Delta)) = S^1$.** The first isomorphism is clear, because the space of unitary automorphisms fixing a vector v is $U(v^\perp)$. The second isomorphism follows from Schwartz lemma. ∎

COROLLARY: Let h be a homogeneous metric on $\Delta = PU(1,1)/S^1$. **Then (Δ, h) is conformally equivalent to $(\Delta$, flat metric$)$.**

Proof: The group $\operatorname{Aut}(\Delta) = PU(1,1)$ acts on Δ holomorphically, that is, preserving the conformal structure of the flat metric. However, homogeneous conformal structure on $PU(1,1)/S^1$ is unique for the same reason the homogeneous metric is unique. ∎

Upper half-plane

REMARK: The map $z \longrightarrow -\sqrt{-1}\,(z-1)^{-1}$ induces a diffeomorphism from the unit disc in $\mathbb{C}$ to the upper half-plane $\mathbb{H}$.

PROPOSITION: The group $\operatorname{Aut}(\Delta)$ acts on the upper half-plane $\mathbb{H}$ as $z \xrightarrow{A} \frac{az+b}{cz+d}$, where $a,b,c,d \in \mathbb{R}$, and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$.

REMARK: The group of such A is naturally identified with $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$.

Proof: The group $PSL(2,\mathbb{R})$ preserves the line $\operatorname{im} z = 0$, hence acts on $\mathbb{H}$ by conformal automorphisms. The stabilizer of a point is S^1 **(prove it).** Now, Lemma 2 implies that $PSL(2,\mathbb{R}) = PU(1,1)$. ■

REMARK: We have shown that $\mathbb{H} = SO(1,2)/S^1$, hence **$\mathbb{H}$ is conformally equivalent to the hyperbolic space.**

Upper half-plane as a Riemannian manifold

DEFINITION: Poincaré half-plane is the upper half-plane equipped with a homogeneous metric of constant negative curvature constructed above.

THEOREM: Let (x, y) be the usual coordinates on the upper half-plane $\mathbb{H}$. **Then the Riemannian structure s on $\mathbb{H}$ is written as $s = const\frac{dx^2+dy^2}{y^2}$.**

Proof: Since the complex structure on $\mathbb{H}$ is the standard one and all Hermitian structures are proportional, we obtain that $s = \mu(dx^2+dy^2)$, where $\mu \in C^\infty(\mathbb{H})$. **It remains to find μ, using the fact that s is $PSL(2,\mathbb{R})$-invariant.**

For each $a \in \mathbb{R}$, the parallel transport $x \longrightarrow x + a$ fixes s, hence μ is a function of y. For any $\lambda \in \mathbb{R}^{>0}$, the map $H_\lambda(x) = \lambda x$ also fixes s; since $\mathbb{H}_\lambda(dx^2+dy^2) = \lambda^2 dx^2 + dy^2$, we have $\mu(\lambda x) = \lambda^{-2}\mu(x)$. ■

Geodesics on Riemannian manifold

DEFINITION: Minimising geodesic in a Riemannian manifold is a piecewise smooth path connecting x to y such that its length is equal to the geodesic distance. **Geodesic** is a piecewise smooth path γ such that for any $x \in \gamma$ there exists a neighbourhood of x in γ which is a minimising geodesic.

EXERCISE: Prove that a big circle in a sphere is a geodesic. Prove that an interval of a big circle of length $\leqslant \pi$ is a minimising geodesic.

Geodesics in Poincaré half-plane

THEOREM: Geodesics on a Poincaré half-plane are vertical straight lines and their images under the action of $SL(2,\mathbb{R})$.

Proof. Step 1: Let $a, b \in \mathbb{H}$ be two points satisfying Re $a =$ Re b, and l the line connecting these two points. Denote by Π the orthogonal projection from $\mathbb{H}$ to the vertical line connecting a to b. For any tangent vector $v \in T_z\mathbb{H}$, one has $|D\pi(v)| \leqslant |v|$, and the equality means that v is vertical (prove it). Therefore, **a projection of a path γ connecting a to b to l has length $\leqslant L(\gamma)$, and the equality is realized only if γ is a straight vertical interval.**

Step 2: For any points a, b in the Poincaré half-plane, **there exists an isometry mapping (a, b) to a pair of points (a_1, b_1) such that Re$(a_1) =$ Re(b_1). (Prove it!)**

Step 3: Using Step 2, we prove that **any geodesic γ on a Poincaré half-plane is obtained as an isometric image of a straight vertical line:** $\gamma = v(\gamma_0)$, $v \in \operatorname{Iso}(\mathbb{H}) = PSL(2,\mathbb{R})$ ∎

Geodesics in Poincaré half-plane

CLAIM: Let S be a circle or a straight line on a complex plane $\mathbb{C} = \mathbb{R}^2$, and S_1 closure of its image in $\mathbb{C}P^1$ inder the natural map $z \longrightarrow 1 : z$. **Then S_1 is a circle, and any circle in $\mathbb{C}P^1$ is obtained this way.**

Proof: The circle $S_r(p)$ of radius r centered in $p \in \mathbb{C}$ is given by equation $|p - z| = r$, in homogeneous coordinates it is $|px - z|^2 = r|x|^2$. This is the zero set of the pseudo-Hermitian form $h(x, z) = |px - z|^2 - |x|^2$, hence it is a circle. ∎

COROLLARY: Geodesics on the Poincaré half-plane are vertical straight lines and half-circles orthogonal to the line $\operatorname{im} z = 0$ in the intersection points.

Proof: We have shown that geodesics in the Poincaré half-plane are Möbius transforms of straight lines orthogonal to $\operatorname{im} z = 0$. However, any Möbius transform preserves angles and maps circles or straight lines to circles or straight lines. ∎