

Riemann surfaces

lecture 7: Kobayashi metric and Arzelà-Ascoli theorem

Misha Verbitsky

Université Libre de Bruxelles

December 1, 2015

Riemannian manifolds (reminder)

DEFINITION: Let $h \in \text{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x, x) > 0$ for any non-zero tangent vector x . Then h is called **Riemannian metric**, of **Riemannian structure**, and (M, h) **Riemannian manifold**.

DEFINITION: For any $x, y \in M$, and any path $\gamma : [a, b] \rightarrow M$ connecting x and y , consider **the length** of γ defined as $L(\gamma) = \int_{\gamma} \left| \frac{d\gamma}{dt} \right| dt$, where $\left| \frac{d\gamma}{dt} \right| = h\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)^{1/2}$. Define **the geodesic distance** as $d(x, y) = \inf_{\gamma} L(\gamma)$, where infimum is taken for all paths connecting x and y .

EXERCISE: Prove that the **geodesic distance satisfies triangle inequality and defines metric on M** .

EXERCISE: Prove that **this metric induces the standard topology on M** .

EXAMPLE: Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x, y) = |x - y|$** .

EXERCISE: Using partition of unity, **prove that any manifold admits a Riemannian structure**.

Conformal structures and almost complex structures (reminder)

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I , let ν be the conformal class of its Hermitian metric. **Then ν is determined by I , and it determines I uniquely.**

DEFINITION: A Riemann surface is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure. A map from one Riemann surface to another is holomorphic if and only if it preserves the conformal structure.

Homogeneous spaces (reminder)

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \rightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\text{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x , or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G , **one has** $M = G/H$, where $H = \text{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \rightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H . ■

REMARK: Let $g(x) = y$. Then $\text{St}_x(G)^g = \text{St}_y(G)$: **all the isotropy groups are conjugate**.

Isotropy representation (reminder)

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\text{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\text{St}_x(G)$ on T_xM .

DEFINITION: A tensor Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant tensor on $\text{St}_x(G)$. For any $y \in M$ obtained as $y = g(x)$, consider the tensor Φ_y on T_yM obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \text{St}_x(G)$. Since Φ is h -invariant, **the tensor Φ_y is independent from the choice of g .**

We proved

THEOREM: Homogeneous tensors on $M = G/H$ are in bijective correspondence with isotropy invariant tensors on T_xM , for any $x \in M$.

Space forms (reminder)

DEFINITION: **Simply connected space form** is a homogeneous manifold of one of the following types:

positive curvature: S^n (an n -dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: \mathbb{R}^n (an n -dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1, n)/SO(n)$, equipped with the natural $SO(1, n)$ -action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

Riemannian metric on space forms (reminder)

LEMMA: Let $G = SO(n)$ act on \mathbb{R}^n in a natural way. **Then there exists a unique G -invariant symmetric 2-form:** the standard Euclidean metric.

Proof: Let g, g' be two G -invariant symmetric 2-forms. Since S^{n-1} is an orbit of G , we have $g(x, x) = g(y, y)$ for any $x, y \in S^{n-1}$. Multiplying g' by a constant, we may assume that $g(x, x) = g'(x, x)$ for any $x \in S^{n-1}$. **Then $g(\lambda x, \lambda x) = g'(\lambda x, \lambda x)$ for any $x \in S^{n-1}, \lambda \in \mathbb{R}$;** however, all vectors can be written as λx . ■

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G -invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■

REMARK: From now on, all space forms are assumed to be homogeneous Riemannian manifolds.

Poincaré-Koebe uniformization theorem (reminder)

DEFINITION: A **Riemannian manifold of constant curvature** is a Riemannian manifold which is locally isometric to a space form.

THEOREM: (Poincaré-Koebe uniformization theorem) Let M be a Riemann surface. **Then M admits a unique complete metric of constant curvature in the same conformal class.**

COROLLARY: **Any Riemann surface is a quotient of a space form X by a discrete group of isometries $\Gamma \subset \text{Iso}(X)$.**

COROLLARY: **Any simply connected Riemann surface is conformally equivalent to a space form.**

REMARK: We shall prove some cases of the uniformization theorem in later lectures.

Möbius transforms (reminder)

DEFINITION: Möbius transform is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: The group $PGL(2, \mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphically.

THEOREM: The natural map from $PGL(2, \mathbb{C})$ to the group $\text{Aut}(\mathbb{C}P^1)$ of Möbius transforms is an isomorphism.

DEFINITION: A circle in S^2 is an orbit of a 1-parametric isometric rotation subgroup $U \subset PGL(2, \mathbb{C})$.

PROPOSITION: The action of $PGL(2, \mathbb{C})$ on $\mathbb{C}P^1$ maps circles to circles.

Proof: Consider a pseudo-Hermitian form h on $V = \mathbb{C}^2$ of signature $(1,1)$. Let h_+ be a positive definite Hermitian form on V . There exists a basis $x, y \in V$ such that $h_+ = \sqrt{-1}x \otimes \bar{x} + \sqrt{-1}y \otimes \bar{y}$ (that is, x, y is orthonormal with respect to h_+) and $h = -\sqrt{-1}\alpha x \otimes \bar{x} + \sqrt{-1}\beta y \otimes \bar{y}$, with $\alpha > 0$, $\beta < 0$ real numbers. Then $\{z \mid h(z, z) = 0\}$ is invariant under the rotation $x, y \longrightarrow x, e^{\sqrt{-1}\theta}y$, hence **it is a circle.** ■

Conformal automorphisms of \mathbb{C} (reminder)

Lemma 1: Let φ be a Möbius transform fixing $\infty \in \mathbb{C}P^1$. **Then** $\varphi(z) = az + b$ **for some** $a, b \in \mathbb{C}$ **and all** $z = z : 1 \in \mathbb{C}P^1$.

THEOREM: (Riemann removable singularity theorem) Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function which is holomorphic outside of a finite set. **Then** f **is holomorphic.**

Proof: Use the Cauchy formula. ■

THEOREM: All conformal automorphisms of \mathbb{C} can be expressed by $z \rightarrow az + b$, where a, b are complex numbers, $a \neq 0$.

Proof: Let φ be a conformal automorphism of \mathbb{C} . The Riemann removable singularity theorem implies that φ **can be extended to a holomorphic automorphism of $\mathbb{C}P^1$** . Indeed, $\mathbb{C}P^1$ is obtained as a 1-point compactification of \mathbb{C} , and any continuous map from \mathbb{C} to \mathbb{C} is extended to a continuous map on $\mathbb{C}P^1$. Now, Lemma 1 implies that $\varphi(z) = az + b$. ■

Schwartz lemma (reminder)

CLAIM: (maximum principle) Let f be a holomorphic function defined on an open set U . **Then f cannot have strict maxima in U . If f has non-strict maxima, it is constant.**

EXERCISE: Prove the maximum principle.

LEMMA: (Schwartz lemma) Let $f : \Delta \rightarrow \Delta$ be a map from disk to itself fixing 0. **Then $|f'(0)| \leq 1$, and equality can be realized only if $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.**

Proof: Consider the function $\varphi := \frac{f(z)}{z}$. Since $f(0) = 0$, it is holomorphic, and since $f(\Delta) \subset \Delta$, on the boundary $\partial\Delta$ we have $|\varphi|_{\partial\Delta} \leq 1$. Now, **the maximum principle implies that $|f'(0)| = |\varphi(0)| \leq 1$** , and equality is realized only if $\varphi = \text{const}$. ■

Conformal automorphisms of the disk (reminder)

CLAIM: Let $\Delta \subset \mathbb{C}$ be the unit disk. **Then the group $\text{Aut}(\Delta)$ of its holomorphic automorphisms acts on Δ transitively.**

Proof: Let $V_a(z) = \frac{z-a}{1-\bar{a}z}$ for some $a \in \Delta$. Then $V_a(0) = -a$. To prove transitivity, it remains to show that $V_a(\Delta) = \Delta$, which is implied from

$$|V_a(z)| = |V_a(z)||z| = \left| \frac{z\bar{z} - a\bar{z}}{1 - \bar{a}z} \right| = \left| \frac{1 - a\bar{z}}{1 - \bar{a}z} \right| = 1.$$

■

REMARK: The group $PU(1, 1) \subset PGL(2, \mathbb{C})$ of unitary matrices preserving a pseudo-Hermitian form h of signature $(1, 1)$ acts on a disk $\{l \in \mathbb{C}P^1 \mid h(l, l) > 0\}$ by holomorphic automorphisms.

COROLLARY: Let $\Delta \subset \mathbb{C}$ be the unit disk, $\text{Aut}(\Delta)$ the group of its conformal automorphisms, and $\Psi : PU(1, 1) \rightarrow \text{Aut}(\Delta)$ the map constructed above. **Then Ψ is an isomorphism.**

COROLLARY: Let h be a homogeneous metric on $\Delta = PU(1, 1)/S^1$. **Then (Δ, h) is conformally equivalent to $(\Delta, \text{flat metric})$.**

Upper half-plane (reminder)

REMARK: The map $z \longrightarrow -\sqrt{-1}(z-1)^{-1}$ induces a diffeomorphism from the unit disc in \mathbb{C} to the upper half-plane \mathbb{H} .

PROPOSITION: The group $\text{Aut}(\Delta)$ acts on the upper half-plane \mathbb{H} as $z \xrightarrow{A} \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$, and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$.

REMARK: The group of such A is naturally identified with $PSL(2, \mathbb{R}) \subset PSL(2, \mathbb{C})$.

REMARK: We have shown that $\mathbb{H} = SO(1, 2)/S^1$, hence \mathbb{H} is conformally equivalent to the hyperbolic space.

Upper half-plane as a Riemannian manifold (reminder)

DEFINITION: Poincaré half-plane is the upper half-plane equipped with a homogeneous metric of constant negative curvature constructed above.

THEOREM: Let (x, y) be the usual coordinates on the upper half-plane \mathbb{H} . Then the Riemannian structure s on \mathbb{H} is written as $s = \text{const} \frac{dx^2 + dy^2}{y^2}$.

Proof: Since the complex structure on \mathbb{H} is the standard one and all Hermitian structures are proportional, we obtain that $s = \mu(dx^2 + dy^2)$, where $\mu \in C^\infty(\mathbb{H})$. It remains to find μ , using the fact that s is $PSL(2, \mathbb{R})$ -invariant.

For each $a \in \mathbb{R}$, the parallel transport $x \rightarrow x + a$ fixes s , hence μ is a function of y . For any $\lambda \in \mathbb{R}^{>0}$, the map $H_\lambda(x) = \lambda x$ also fixes s ; since $\mathbb{H}_\lambda(dx^2 + dy^2) = \lambda^2 dx^2 + dy^2$, we have $\mu(\lambda x) = \lambda^{-2} \mu(x)$. ■

Geodesics on Riemannian manifold (reminder)

THEOREM: Geodesics on a Poincaré half-plane are vertical straight lines and their images under the action of $SL(2, \mathbb{R})$.

CLAIM: Let S be a circle or a straight line on a complex plane $\mathbb{C} = \mathbb{R}^2$, and S_1 closure of its image in $\mathbb{C}P^1$ under the natural map $z \rightarrow 1 : z$. **Then S_1 is a circle, and any circle in $\mathbb{C}P^1$ is obtained this way.**

Proof: The circle $S_r(p)$ of radius r centered in $p \in \mathbb{C}$ is given by equation $|p - z| = r$, in homogeneous coordinates it is $|px - z|^2 = r|x|^2$. This is the zero set of the pseudo-Hermitian form $h(x, z) = |px - z|^2 - |x|^2$, hence it is a circle.

■

COROLLARY: Geodesics on the Poincaré half-plane are vertical straight lines and half-circles orthogonal to the line $\text{im } z = 0$ in the intersection points.

Proof: We have shown that geodesics in the Poincaré half-plane are Möbius transforms of straight lines orthogonal to $\text{im } z = 0$. However, any Möbius transform preserves angles and maps circles or straight lines to circles or straight lines. ■

Poincaré metric on disk

DEFINITION: Poincaré metric on a unit disk $\Delta \subset \mathbb{C}$ is an $\text{Aut}(\Delta)$ -invariant metric (it is unique up to a constant multiplier).

DEFINITION: Let $f : M \rightarrow M_1$ be a map of metric spaces. Then f is called **C -Lipschitz** if $d(x, y) \geq C d(f(x), f(y))$. A map is called **Lipschitz** if it is C -Lipschitz for some $C > 0$.

THEOREM: (Schwartz-Pick lemma)

Any holomorphic map $\varphi : \Delta \rightarrow \Delta$ from a unit disk to itself is 1-Lipschitz with respect to Poincaré metric.

Proof. Step 1: We need to prove that for each $x \in \Delta$ the norm of the differential satisfies $|D\varphi_x| \leq 1$. Since the automorphism group acts on Δ transitively, **it suffices to prove that $|D\varphi_x| \leq 1$ when $x = 0$ and $\varphi(x) = 0$.**

Step 2: This is Schwartz lemma. ■

Kobayashi pseudometric

DEFINITION: Pseudometric on M is a function $d : M \times M \rightarrow \mathbb{R}^{\geq 0}$ which is symmetric: $d(x, y) = d(y, x)$ and satisfies the triangle inequality $d(x, y) + d(y, z) \geq d(x, z)$.

REMARK: Let \mathfrak{D} be a set of pseudometrics. **Then** $d_{\max}(x, y) := \sup_{d \in \mathfrak{D}} d(x, y)$ **is also a pseudometric.**

DEFINITION: The **Kobayashi pseudometric** on a complex manifold M is d_{\max} for the set \mathfrak{D} of all pseudometrics such that any holomorphic map from the Poincaré disk to M is distance-decreasing.

EXERCISE: Prove that **the distance between points x, y in Kobayashi pseudometric is infimum of the Poincaré distance over all sets of Poincaré disks connecting x to y .**

EXAMPLE: The Kobayashi pseudometric on \mathbb{C} vanishes.

CLAIM: Any holomorphic map $X \xrightarrow{\varphi} Y$ is **1-Lipschitz with respect to the Kobayashi pseudometric.**

Proof: If $x \in X$ is connected to x' by a sequence of Poincaré disks $\Delta_1, \dots, \Delta_n$, then $\varphi(x)$ is connected to $\varphi(x')$ by $\varphi(\Delta_1), \dots, \varphi(\Delta_n)$. ■

Kobayashi hyperbolic manifolds

COROLLARY: Let $B \subset \mathbb{C}^n$ be a unit ball, and $x, y \in B$ points with coordinates $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$. Since x_i, y_i belongs to Δ , it makes sense to compute the Poincare distance $d_P(x_i, y_i)$. **Then $d_K(x, y) \geq \max_i d_P(x_i, y_i)$.**

Proof: Each of projection maps $\Pi_i : B \rightarrow \Delta$ is 1-Lipshitz. ■

DEFINITION: A variety is called **Kobayashi hyperbolic** if the Kobayashi pseudometric d_K is non-degenerate.

DEFINITION: A **domain** in \mathbb{C}^n is an open subset. A **bounded domain** is an open subset contained in a ball.

COROLLARY: **Any bounded domain Ω in \mathbb{C}^n is Kobayashi hyperbolic.**

Proof: Without restricting generality, we may assume that $\Omega \subset B$ where B is an open ball. Then the Kobayashi distance in Ω is \geq that in B . However, the Kobayashi distance in B is bounded by the metric $d(x, y) := \max_i d_P(x_i, y_i)$ as follows from above. ■

Caratheodory metric

DEFINITION: Let $x, y \in M$ be points on a complex manifold. Define **Caratheodory pseudometric** as $d_C(x, y) = \sup\{d_P(f(x), f(y))\}$, where the supremum is taken over all holomorphic map $f : M \rightarrow \Delta$, and d_P is Poincare metric on the disk Δ .

REMARK: Usually the term “Kobayashi/Caratheodory pseudometric” is abbreviated to “Kobayashi/Caratheodory metric”, **even when it is not a metric.**

REMARK: Caratheodory pseudometric **satisfies the triangle inequality** because a supremum of pseudometrics satisfies triangle inequality.

EXERCISE: Prove that **Caratheodory pseudometric is bounded by the Kobayashi pseudometric:** $d_K \geq d_C$.

REMARK: Clearly, $d_C \neq 0$ on any bounded domain.

Complex hyperbolic space

DEFINITION: Let $V = \mathbb{C}^{n+1}$ be a complex vector space equipped with a Hermitian metric h of signature $(1, n)$, and $\mathbb{H}_{\mathbb{C}}^n \subset \mathbb{P}V$ projectivization of the set of positive vectors $\{x \in V \mid h(x, \bar{x}) > 0\}$. Then $\mathbb{H}_{\mathbb{C}}^n$ is equipped with a homogeneous action of $U(1, n)$. The same argument as used for space forms implies that $\mathbb{H}_{\mathbb{C}}^n$ admits a $U(1, n)$ -invariant Hermitian metric, which is unique up to a constant multiplier. This Hermitian complex manifold is called **complex hyperbolic space**.

REMARK: For $n > 1$ it is not isometric, to the real hyperbolic spaces defined earlier.

REMARK: As a complex manifold $\mathbb{H}_{\mathbb{C}}^n$ is isomorphic to an open ball in \mathbb{C}^n .

REMARK: The Kobayashi metric and the Caratheodory metric on $\mathbb{H}_{\mathbb{C}}^n$ are $U(1, n)$ -invariant, because $U(1, n)$ acts holomorphically, hence proportional to the hyperbolic metric, which is also called **Bergman metric** on an open ball.

EXERCISE: Prove that **Kobayashi metric on a ball in \mathbb{C}^n is equal to the Caratheodory metric**.

Uniform convergence for Lipschitz maps

DEFINITION: A sequence of maps $f_i : M \rightarrow N$ between metric spaces **uniformly converges** (or **converges uniformly on compacts**) to $f : M \rightarrow N$ if for any compact $K \subset M$, we have $\lim_{i \rightarrow \infty} \sup_{x \in K} d(f_i(x), f(x)) = 0$.

Claim 1: Suppose that a sequence $f_i : M \rightarrow N$ of 1-Lipschitz maps converges to f pointwise in a countable dense subset $M' \subset M$. **Then f_i converges to f uniformly on compacts.**

Proof: Let $K \subset M$ be a compact set, and $N_\varepsilon \subset M'$ a finite subset such that K is a union of ε -balls centered in N_ε (such N_ε is called **an ε -net**). Then there exists N such that $\sup_{x \in N_\varepsilon} d(f_{N+i}(x), f(x)) < \varepsilon$ for all $i \geq 0$. Since f_i are 1-Lipschitz, this implies that

$$\begin{aligned} \sup_{y \in K} d(f_{N+i}(y), f(y)) &\leq \\ &\leq \sup_{x \in N_\varepsilon} d(f_{N+i}(x), f(x)) + \inf_{x \in N_\varepsilon} (d(f_{N+i}(x), y) + d(f(x), y)) \leq 3\varepsilon. \end{aligned}$$

■

EXERCISE: Prove that the limit f is also 1-Lipschitz.

REMARK: This proof works when M is a pseudo-metric space, as long as N is a metric space.

Arzela-Ascoli theorem for Lipschitz maps

DEFINITION: Let M, N be metric spaces. A subset $B \subset M$ is **bounded** if it is contained in a ball. A family $\{f_\alpha\}$ of functions $f_\alpha : M \rightarrow N$ is called **uniformly bounded on compacts** if for any compact subset $K \subset M$, there is a bounded subset $C_K \subset N$ such that $f_\alpha(K) \subset C_K$ for any element f_α of the family.

THEOREM: (Arzela-Ascoli for Lipschitz maps)

Let $\mathcal{F} := \{f_\alpha\}$ be an infinite uniformly bounded set of 1-Lipschitz maps $f_\alpha : M \rightarrow \mathbb{C}$, where M is a pseudo-metric space. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \rightarrow \mathbb{C}$ uniformly.**

REMARK: The limit f is clearly also 1-Lipschitz.

Proof. Step 1: Suppose we can prove Arzela-Ascoli when M is compact. Then we can choose a sequence of compact subsets $K_i \subset M$, find subsequences in \mathcal{F} converging on each K_i , and use the diagonal method to find a subsequence converging on all K_i . Therefore, **we can assume that M is bounded, and all maps $f_\alpha : M \rightarrow \mathbb{C}$ map M into a compact subset $N \subset \mathbb{C}$.**

Arzela-Ascoli theorem for Lipschitz maps (2)

Proof. Step 1: We can assume that M is compact, and all maps $f_\alpha : M \rightarrow \mathbb{C}$ map M into a compact subset $N \subset \mathbb{C}$.

Step 2. Consider the set of maps $\text{Map}(M, N)$ with the **Tychonoff topology**, that is, identified with an infinite product of M copies of N . **By Tychonoff theorem, $\text{Map}(M, N)$ is compact.**

Step 3. Suppose that each point of $\text{Map}(M, N)$ has a neighbourhood containing only finitely many elements of \mathcal{F} . Using compactness, we can cover $\text{Map}(M, N)$ by finitely many of such neighbourhoods. Then \mathcal{F} is finite, which is a contradiction. Choose $f \in \text{Map}(M, N)$ such that in each neighbourhood of f there are infinitely many elements of \mathcal{F} . Such f is **a limit point of \mathcal{F} in Tychonoff topology**, which is the same as topology of pointwise convergence.

Step 4. By definition of pointwise convergence, for any finite set $S \subset M$, there exists a subsequence f_i of \mathcal{F} which converges to $f \in \text{Map}(M, N)$ in S . Using diagonal method, we choose a subsequence f_i of \mathcal{F} which converges to $f \in \text{Map}(M, N)$ pointwise in a dense countable set. **Then f_i converges to f uniformly by Claim 1. ■**

Normal families of holomorphic functions

DEFINITION: Let M be a complex manifold. A family $\mathcal{F} := \{f_\alpha\}$ of holomorphic functions $f_\alpha : M \rightarrow \mathbb{C}$ is called **normal family** if \mathcal{F} is uniformly bounded on compact subsets.

THEOREM: (Montel's theorem)

Let M be a complex manifold with countable base, and \mathcal{F} a normal, infinite family of holomorphic functions. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \rightarrow \mathbb{C}$ uniformly**, and f is holomorphic.

Proof. Step 1: As in the first step of Arzela-Ascoli, it suffices to prove Montel's theorem on a subset of M where \mathcal{F} is bounded. Therefore, **we may assume that all f_α map M into a disk Δ .**

Step 2: All f_α are 1-Lipschitz with respect to Kobayashi metric. Therefore, **Arzela-Ascoli theorem can be applied, giving a uniform limit $f = \lim f_i$.**

Step 3: A uniform limit of holomorphic functions is holomorphic by Cauchy formula. ■

REMARK: The sequence $f = \lim f_i$ **converges uniformly with all derivatives**, again by Cauchy formula.

Riemann mapping theorem

Next lecture, I will prove the Riemann mapping theorem, which is one of the simplest uniformization theorems.

THEOREM: Let $\Omega \subset \Delta$ be a simply connected domain. **Then Ω is biholomorphic to Δ .**

Idea of a proof: We consider the Kobayashi metric on Ω and Δ , and let \mathcal{F} be the set of all injective holomorphic maps $\Omega \rightarrow \Delta$. Consider $x \in \Omega$, and let f be a map with $|df_x|$ maximal in the sense of Kobayashi metric. **Such f exists by Montel's theorem. We prove that f is an isometry**, and hence biholomorphic.