Riemann surfaces

lecture 7: Riemann mapping theorem

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Riemannian manifolds (reminder)

DEFINITION: Let $h \in \mathrm{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: For any $x.y \in M$, and any path $\gamma: [a,b] \longrightarrow M$ connecting x and y, consider **the length** of γ defined as $L(\gamma) = \int_\gamma |\frac{d\gamma}{dt}| dt$, where $|\frac{d\gamma}{dt}| = h(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2}$. Define **the geodesic distance** as $d(x,y) = \inf_\gamma L(\gamma)$, where infimum is taken for all paths connecting x and y.

EXERCISE: Prove that the **geodesic distance satisfies triangle inequality and defines metric on M.**

EXERCISE: Prove that **this metric induces the standard topology on M.**

EXAMPLE: Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x,y) = |x-y|$.**

EXERCISE: Using partition of unity, **prove that any manifold admits a Riemannian structure.**

Conformal structures and almost complex structures (reminder)

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I, let ν be the conformal class of its Hermitian metric. **Then ν is determined by I, and it determines I uniquely.**

DEFINITION: A Riemann surface is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure. A map from one Riemann surface to another is holomorphic if and only if it preserves the conformal structure.

Homogeneous spaces (reminder)

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\mathrm{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \mathrm{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\mathrm{St}_x(G)^g = \mathrm{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation (reminder)

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\operatorname{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\operatorname{St}_x(G)$ on T_xM.

DEFINITION: A tensor Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant tensor on $\operatorname{St}_x(G)$. For any $y \in M$ obtained as $y = g(x)$, consider the tensor Φ_y on T_yM obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \operatorname{St}_x(G)$. Since Φ is h-invariant, **the tensor Φ_y is independent from the choice of g.**

We proved

THEOREM: Homogeneous tensors on $M = G/H$ are in bijective correspondence with isotropy invariant tensors on T_xM, for any $x \in M$.

Space forms (reminder)

DEFINITION: Simply connected space form is a homogeneous manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

Riemannian metric on space forms (reminder)

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique G-invariant symmetric 2-form:** the standard Euclidean metric.

Proof: Let g, g' be two G-invariant symmetric 2-forms. Since S^{n-1} is an orbit of G, we have $g(x,x) = g(y,y)$ for any $x, y \in S^{n-1}$. Multiplying g' by a constant, we may assume that $g(x,x) = g'(x,x)$ for any $x \in S^{n-1}$. **Then $g(\lambda x, \lambda x) = g'(\lambda x, \lambda x)$ for any $x \in S^{n-1}$, $\lambda \in \mathbb{R}$;** however, all vectors can be written as λx. ■

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■

REMARK: From now on, all space forms are assumed to be homogeneous Riemannian manifolds.

Poincaré-Koebe uniformization theorem (reminder)

DEFINITION: A **Riemannian manifold of constant curvature** is a Riemannian manifold which is locally isometric to a space form.

THEOREM: (Poincaré-Koebe uniformization theorem) Let M be a Riemann surface. **Then M admits a unique complete metric of constant curvature in the same conformal class.**

COROLLARY: Any Riemann surface is a quotient of a space form X by a discrete group of isometries $\Gamma \subset \operatorname{Iso}(X)$.

COROLLARY: Any simply connected Riemann surface is conformally equivalent to a space form.

REMARK: We shall prove some cases of the uniformization theorem in later lectures.

Möbius transforms (reminder)

DEFINITION: Möbius transform is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: The group $PGL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphially.

THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group $\operatorname{Aut}(\mathbb{C}P^1)$ of Möbius transforms is an isomorphism.

DEFINITION: A circle in S^2 is an orbit of a 1-parametric isometric rotation subgroup $U \subset PGL(2,\mathbb{C})$.

PROPOSITION: The action of $PGL(2,\mathbb{C})$ on $\mathbb{C}P^1$ maps circles to circles.

Proof: Consider a pseudo-Hermitian form h on $V = \mathbb{C}^2$ of signature (1,1). Let h_+ be a positive definite Hermitian form on V. There exists a basis $x, y \in V$ such that $h_+ = \sqrt{-1}\, x \otimes \overline{x} + \sqrt{-1}\, y \otimes \overline{y}$ (that is, x, y is orthonormal with respect to h_+) and $h = -\sqrt{-1}\, \alpha x \otimes \overline{x} + \sqrt{-1}\, \beta y \otimes \overline{y}$, with $\alpha > 0$, $\beta < 0$ real numbers. Then $\{z \mid h(z,z) = 0\}$ is invariant under the rotation $x, y \longrightarrow x, e^{\sqrt{-1}\,\theta} y$, hence **it is a circle.** ■

Conformal automorphisms of $\mathbb{C}$ (reminder)

Lemma 1: Let φ be a Möbius transform fixing $\infty \in \mathbb{C}P^1$. **Then $\varphi(z) = az + b$ for some $a, b \in \mathbb{C}$ and all $z = z : 1 \in \mathbb{C}P^1$.**

THEOREM: (Riemann removable singularity theorem) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function which is holomorphic outside of a finite set. **Then f is holomorphic.**

Proof: Use the Cauchy formula. ■

THEOREM: All conformal automorphisms of $\mathbb{C}$ can be expressed by $z \longrightarrow az + b$, where a, b are complex numbers, $a \neq 0$.

Proof: Let φ be a conformal automorphism of $\mathbb{C}$. The Riemann removable singularity theorem implies that **φ can be extended to a holomorphic automorphism of $\mathbb{C}P^1$.** Indeed, $\mathbb{C}P^1$ is obtained as a 1-point compactification of $\mathbb{C}$, and any continuous map from $\mathbb{C}$ to $\mathbb{C}$ is extended to a continuous map on $\mathbb{C}P^1$. Now, Lemma 1 implies that $\varphi(z) = az + b$. ■

Schwartz lemma (reminder)

CLAIM: (maximum principle) Let f be a holomorphic function defined on an open set U. **Then f cannot have strict maxima in U. If f has non-strict maxima, it is constant.**

EXERCISE: Prove the maximum principle.

LEMMA: (Schwartz lemma) Let $f: \Delta \longrightarrow \Delta$ be a map from disk to itself fixing 0. **Then $|f'(0)| \leqslant 1$, and equality can be realized only if $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.**

Proof: Consider the function $\varphi := \frac{f(z)}{z}$. Since $f(0) = 0$, it is holomorphic, and since $f(\Delta) \subset \Delta$, on the boundary $\partial\Delta$ we have $|\varphi||_{\partial\Delta} \leqslant 1$. Now, **the maximum principle implies that $|f'(0)| = |\varphi(0)| \leqslant 1$,** and equality is realized only if $\varphi = const$. ■

Conformal automorphisms of the disk (reminder)

CLAIM: Let $\Delta \subset \mathbb{C}$ be the unit disk. **Then the group** $\operatorname{Aut}(\Delta)$ **of its holomorphic automorphisms acts on** Δ **transitively.**

Proof: Let $V_a(z) = \frac{z-a}{1-\overline{a}z}$ for some $a \in \Delta$. Then $V_a(0) = -a$. To prove transitivity, it remains to show that $V_a(\Delta) = \Delta$, which is implied from

$$|V_a(z)| = |V_a(z)||z| = \left|\frac{z\overline{z} - a\overline{z}}{1 - \overline{a}z}\right| = \left|\frac{1 - a\overline{z}}{1 - \overline{a}z}\right| = 1.$$

■

REMARK: The group $PU(1,1) \subset PGL(2,\mathbb{C})$ of unitary matrices preserving a pseudo-Hermitian form h of signature (1,1) acts on a disk $\{l \in \mathbb{C}P^1 \ \mid \ h(l,l) > 0\}$ by holomorphic automorphisms.

COROLLARY: Let $\Delta \subset \mathbb{C}$ be the unit disk, $\operatorname{Aut}(\Delta)$ the group of its conformal automorphisms, and $\Psi: \ PU(1,1) \longrightarrow \operatorname{Aut}(\Delta)$ the map constructed above. **Then Ψ is an isomorphism.**

COROLLARY: Let h be a homogeneous metric on $\Delta = PU(1,1)/S^1$. **Then** (Δ, h) **is conformally equivalent to** $(\Delta,$ **flat metric).**

Upper half-plane (reminder)

REMARK: The map $z \longrightarrow -\sqrt{-1}\,(z-1)^{-1}$ induces a diffeomorphism from the unit disc in $\mathbb{C}$ to the upper half-plane $\mathbb{H}$.

PROPOSITION: The group $\operatorname{Aut}(\Delta)$ acts on the upper half-plane $\mathbb{H}$ as $z \xrightarrow{A} \frac{az+b}{cz+d}$, where $a,b,c,d \in \mathbb{R}$, and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$.

REMARK: The group of such A is naturally identified with $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$.

REMARK: We have shown that $\mathbb{H} = SO(1,2)/S^1$, hence **$\mathbb{H}$ is conformally equivalent to the hyperbolic space.**

Upper half-plane as a Riemannian manifold (reminder)

DEFINITION: Poincaré half-plane is the upper half-plane equipped with a homogeneous metric of constant negative curvature constructed above.

THEOREM: Let (x, y) be the usual coordinates on the upper half-plane $\mathbb{H}$. **Then the Riemannian structure s on $\mathbb{H}$ is written as $s = const\frac{dx^2+dy^2}{y^2}$.**

Proof: Since the complex structure on $\mathbb{H}$ is the standard one and all Hermitian structures are proportional, we obtain that $s = \mu(dx^2+dy^2)$, where $\mu \in C^\infty(\mathbb{H})$. **It remains to find μ, using the fact that s is $PSL(2,\mathbb{R})$-invariant.**

For each $a \in \mathbb{R}$, the parallel transport $x \longrightarrow x + a$ fixes s, hence μ is a function of y. For any $\lambda \in \mathbb{R}^{>0}$, the map $H_\lambda(x) = \lambda x$ also fixes s; since $\mathbb{H}_\lambda(dx^2+dy^2) = \lambda^2 dx^2 + dy^2$, we have $\mu(\lambda x) = \lambda^{-2}\mu(x)$. ■

Geodesics on Riemannian manifold (reminder)

THEOREM: Geodesics on a Poincaré half-plane are vertical straight lines and their images under the action of $SL(2, \mathbb{R})$.

CLAIM: Let S be a circle or a straight line on a complex plane $\mathbb{C} = \mathbb{R}^2$, and S_1 closure of its image in $\mathbb{C}P^1$ inder the natural map $z \longrightarrow 1 : z$. **Then S_1 is a circle, and any circle in $\mathbb{C}P^1$ is obtained this way.**

Proof: The circle $S_r(p)$ of radius r centered in $p \in \mathbb{C}$ is given by equation $|p - z| = r$, in homogeneous coordinates it is $|px - z|^2 = r|x|^2$. This is the zero set of the pseudo-Hermitian form $h(x, z) = |px - z|^2 - |x|^2$, hence it is a circle. ■

COROLLARY: Geodesics on the Poincaré half-plane are vertical straight lines and half-circles orthogonal to the line $\operatorname{im} z = 0$ in the intersection points.

Proof: We have shown that geodesics in the Poincaré half-plane are Möbius transforms of straight lines orthogonal to $\operatorname{im} z = 0$. However, any Möbius transform preserves angles and maps circles or straight lines to circles or straight lines. ■

Poincaré metric on disk (reminder)

DEFINITION: Poincaré metric on a unit disk $\Delta \subset \mathbb{C}$ is an Aut(Δ)-invariant metric **(it is unique up to a constant multiplier)**.

DEFINITION: Let $f : M \longrightarrow M_1$ be a map of metric spaces. Then f is called **C-Lipschitz** if $d(x,y) \geqslant Cd(f(x), f(y))$. A map is called **Lipschitz** if it is C-Lipschitz for some $C > 0$.

THEOREM: (Schwartz-Pick lemma)
Any holomorphic map $\varphi : \Delta \longrightarrow \Delta$ from a unit disk to itself is 1-Lipschitz with respect to Poicaré metric.

Proof. Step 1: We need to prove that for each $x \in \Delta$ the norm of the differential satisfies $|D\varphi_x| \leqslant 1$. Since the automorphism group acts on Δ transitively, **it suffices to prove that $|D\varphi_x| \leqslant 1$ when $x = 0$ and $\varphi(x) = 0$.**

Step 2: This is Schwartz lemma. ■

Kobayashi pseudometric (reminder)

DEFINITION: Pseudometric on M is a function $d: M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric: $d(x,y) = d(y,x)$ and satisfies the triangle inequality $d(x,y) + d(y,z) \geqslant d(x,z)$.

REMARK: Let $\mathfrak{D}$ be a set of pseudometrics. **Then $d_{\max}(x,y) := \sup_{d \in \mathfrak{D}} d(x,y)$ is also a pseudometric.**

DEFINITION: The **Kobayashi pseudometric** on a complex manifold M is $d_{\max}$ for the set $\mathfrak{D}$ of all pseudometrics such that any holomorphic map from the Poincaré disk to M is distance-decreasing.

EXERCISE: Prove that **the distance between points x, y in Kobayashi pseudometric is infimum of the Poincaré distance over all sets of Poincaré disks connecting x to y.**

EXERCISE: Prove that the Kobayashi pseudometric on $\mathbb{C}$ vanishes.

CLAIM: Any holomorphic map $X \xrightarrow{\varphi} Y$ is 1-Lipschitz with respect to the Kobayashi pseudometric.

Proof: If $x \in X$ is connected to x' by a sequence of Poincare disks $\Delta_1, ..., \Delta_n$, then $\varphi(x)$ is connected to $\varphi(x')$ by $\varphi(\Delta_1), ..., \varphi(\Delta_n)$. ■

Kobayashi hyperbolic manifolds (reminder)

COROLLARY: Let $B \subset \mathbb{C}^n$ be a unit ball, and $x, y \in B$ points with coordinates $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$. Since x_i, y_i belongs to Δ, it makes sense to compute the Poincare distance $d_P(x_i, y_i)$. **Then $d_K(x,y) \geqslant \max_i d_P(x_i, y_i)$.**

Proof: Each of projection maps $\Pi_i : B \longrightarrow \Delta$ is 1-Lipshitz. ■

DEFINITION: A variety is called **Kobayashi hyperbolic** if the Kobayashi pseudometric d_K is non-degenerate.

DEFINITION: A **domain** in $\mathbb{C}^n$ is an open subset. A **bounded domain** is an open subset contained in a ball.

COROLLARY: Any bounded domain Ω in $\mathbb{C}^n$ is Kobayashi hyperbolic.

Proof: Without restricting generality, we may assume that $\Omega \subset B$ where B is an open ball. Then the Kobayashi distance in Ω is $\geqslant$ that in B. However, the Kobayashi distance in B is bounded by the metric $d(x,y) := \max_i d_P(x_i, y_i)$ as follows from above. ■

Uniform convergence for Lipschitz maps (reminder)

DEFINITION: A sequence of maps $f_i : M \longrightarrow N$ between metric spaces **uniformly converges** (or **converges uniformly on compacts**) to $f : M \longrightarrow N$ if for any compact $K \subset M$, we have $\lim_{i\to\infty} \sup_{x\in K} d(f_i(x), f(x)) = 0$.

Claim 1: Suppose that a sequence $f_i : M \longrightarrow N$ of 1-Lipschitz maps converges to f pointwise in a countable dense subset $M' \subset M$. **Then f_i converges to f uniformly on compacts.**

Proof: Let $K \subset M$ be a compact set, and $N_\varepsilon \subset M'$ a finite subset such that K is a union of ε-balls centered in N_ε (such N_ε is called **an ε-net**). Then there exists N such that $\sup_{x\in N_\varepsilon} d(f_{N+i}(x), f(x)) < \varepsilon$ for all $i \geqslant 0$. Since f_i are 1-Lipschitz, this implies that

$$\sup_{y\in K} d(f_{N+i}(y), f(y) \leqslant$$
$$\leqslant \sup_{x\in N_\varepsilon} d(f_{N+i}(x), f(x)) + \inf_{x\in N_\varepsilon} (d(f_{N+i}(x), y) + d(f(x), y)) \leqslant 3\varepsilon.$$

■

EXERCISE: Prove that the limit f is also 1-Lipschitz.

REMARK: This proof works when M is a pseudo-metric space, as long as N is a metric space.

Arzela-Ascoli theorem for Lipschitz maps (reminder)

DEFINITION: Let M, N be metric spaces. A subset $B\subset M$ is **bounded** if it is contained in a ball. A family $\{f_\alpha\}$ of functions $f_\alpha: M\longrightarrow N$ is called **uniformly bounded on compacts** if for any compact subset $K\subset M$, there is a bounded subset $C_K\subset N$ such that $f_\alpha(K)\subset C_K$ for any element f_α of the family.

THEOREM: (Arzela-Ascoli for Lipschitz maps)
Let $\mathcal{F} := \{f_\alpha\}$ be an infinite uniformly bounded set of 1-Lipschitz maps $f_\alpha : M\longrightarrow\mathbb{C}$, where M is a pseudo-metric space. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\}\subset\mathcal{F}$ which converges to $f: M\longrightarrow\mathbb{C}$ uniformly.**

REMARK: The limit f is clearly also 1-Lipschitz.

Normal families of holomorphic functions (reminder)

DEFINITION: Let M be a complex manifold. A family $\mathcal{F} := \{f_\alpha\}$ of holomorphic functions $f_\alpha : M \longrightarrow \mathbb{C}$ is called **normal family** if $\mathcal{F}$ is uniformly bounded on compact subsets.

THEOREM: (Montel's theorem)
Let M be a complex manifold with countable base, and $\mathcal{F}$ a normal, infinite family of holomorphic functions. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \longrightarrow \mathbb{C}$ uniformly,** and f is holomorphic.

Proof. Step 1: As in the first step of Arzela-Ascoli, it suffices to prove Montel's theorem on a subset of M where $\mathcal{F}$ is bounded. Therefore, **we may assume that all f_α map M into a disk Δ.**

Step 2: All f_α are 1-Lipschitz with respect to Kobayashi metric. Therefore, **Arzela-Ascoli theorem can be applied, giving a uniform limit $f = \lim f_i$.**

Step 3: A uniform limit of holomorphic functions is holomorphic by Cauchy formula. ■

REMARK: The sequence $f = \lim f_i$ converges uniformly with all derivatives, again by Cauchy formula.

Riemann mapping theorem

THEOREM: Let $\Omega \subset \Delta$ be a simply connected domain. **Then Ω is biholomorphic to Δ.**

Idea of a proof: We consider the Kobayashi metric on Ω and Δ, and let $\mathcal{F}$ be the set of all injective holomorphic maps $\Omega \longrightarrow \Delta$. Consider $x \in \Omega$, and let f be a map with $|df_x|$ maximal in the sense of Kobayashi metric. **Such f exists by Montel's theorem. We prove that f is an isometry**, and hence biholomorphic.

PROPOSITION: Let $\mathcal{H}$ be the set of holomorphic maps $f: \Omega_1 \longrightarrow \Omega_2$ between Riemann surfaces, equipped with uniform topology, and $\mathcal{H}_0$ its subset consisting of injective maps. **Then $\mathcal{H}_0$ is closed in $\mathcal{H}$.**

Proof: Let f_i be a sequence of injective maps converging to $f: \Omega_1 \longrightarrow \Omega_2$ which is not injective. Then $f(a) = f(b)$ for some $a \neq b$ in Ω_1. Choose open disks A and B containing a and b. Then the Proposition is implied by the following lemma.

LEMMA: Let $\mathcal{R}$ be the set of all pairs of distinct holomorphic functions $f, g: \Delta \longrightarrow \mathbb{C}$ continuously extended to the boundary such that $f(x) = g(x)$ for some $x \in \Delta$. Then $\mathcal{R}$ is open in uniform topology.

The set of non-injective maps is open

LEMMA: Let $\mathcal{R}$ be the set of all pairs of distinct holomorphic functions $f, g: \Delta \longrightarrow \mathbb{C}$ continuously extended to the boundary such that $f(x) = g(x)$ for some $x \in \Delta$. Then $\mathcal{R}$ is open in uniform topology.

Proof. Step 1: The set of all points $z \in \Delta$ where $f = g$ is discrete **(prove it).** This implies that **we can replace Δ by a smaller disc containing x such that $f \neq g$ everywhere on its boundary.**

Step 2: Consider the function $\alpha\frac{(f-g)'}{f-g}$ on Δ. **This function has a simple pole in all the points where $f = g$.** Therefore, $n_{f,g} := \frac{1}{\pi\sqrt{-1}} \int_{\partial\Delta} \alpha dz$ is equal to the number of points $x \in \Delta$ such that $f(x) = g(x)$.

Step 3: Since the integral is continuous in unform topology, **this number is locally constant on the space of pairs such $f, g: \Delta \longrightarrow \mathbb{C}$.** Therefore, the set $\mathcal{R}$ of all f, g with $n_{f,g} \neq 0$ is open. ∎

Coverings

DEFINITION: Constant sheaf with coefficients in a set S is a sheaf $\mathcal{F}$ such that for any $U \subset M$, the space of sections of $\mathcal{F}$ is S, independent on U. **Locally constant sheaf** is a sheaf $\mathcal{F}$ such that each point $x \in M$ has a neighbourhood $U \subset M$ such that the restriction of $\mathcal{F}$ to U is constant.

DEFINITION: A continuous map $\pi : \tilde{M} \longrightarrow M$ of topological spaces is a **covering** if π is locally a diffeomorphism, and the space of sections of π is a locally constant sheaf.

EXAMPLE: The map $x \longrightarrow x^2$ is a covering from $\mathbb{C}^* := \mathbb{C}\backslash 0$ to itself **(prove it)**.

Homotopy lifting principle

DEFINITION: A topological space X is **locally path connected** if for each $x \in X$ and each neighbourhood $U \ni x$, there exists a smaller neighbourhood $W \ni x$ which is path connected.

THEOREM: (homotopy lifting principle)
Let X be a simply connected, locally path connected topological space, and $\tilde{M} \longrightarrow M$ a covering map. Then for each continuous map $X \longrightarrow M$, there exists a lifting $X \longrightarrow \tilde{M}$ making the following diagram commutative.

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \tilde{M} \arrow[d] \\ & M \end{tikzcd}$$

COROLLARY: Let $\varphi: \Omega \longrightarrow \mathbb{C}^*$ be a holomorphic map from a simply connected domain Ω. **Then there exists a holomorphic map $\varphi_1: \Omega \longrightarrow \mathbb{C}^*$ such that for all $z \in \Delta$, $\varphi(z) = \varphi_1(z)^2$.**

Proof: We apply homotopy lifting principle to $X = \Omega$, $M = \tilde{M} = \mathbb{C}^*$, and $\tilde{M} \longrightarrow M$ mapping x to x^2. ■

REMARK: We denote $\varphi_1(z)$ by $\sqrt{\varphi(z)}$, for obvious reasons.

Kobayashi metric and the map $x \longrightarrow x^2$

CLAIM: Consider a non-bijective holomorphic map $\varphi: \Delta \longrightarrow \Delta$ from Poincare disk to itself. **Then $|d\varphi| < 1$ at each point,** where $d\varphi$ is a norm of an operator $d\varphi: T_x\Delta \longrightarrow T_{\varphi(x)}\Delta$ taken with respect to the Poincare metric.

Proof: Let $\varphi: \Delta \longrightarrow \Delta$ be a holomorphic map which satisfies $|d\varphi| = 1$ at $x \in \Delta$. Replacing φ by $\gamma_1 \circ \varphi \circ \gamma_2$ if necessary, where γ_i are biholomorphic isometries of Δ, we may assume that $x = 0$ and $\varphi(x) = 0$. By Schwartz lemma for such φ, $|d\varphi(0)| = 1$ implies that φ is a linear biholomorphic map. ■

Corollary 1: Let $\varphi: \Delta \longrightarrow \Delta\backslash 0$ be a holomorphic function, and $\sqrt{\varphi}$ a holomorphic function defined above. Let $|d\varphi|(x)$ denote the norm of the operator $d\varphi$ at $x \in \Delta$ computed with respect to the Poincare metric on Δ. **Then $|d\varphi|(x) < |d\sqrt{\varphi}|(x)$ for any $x \in \Delta$.**

Proof: Let $\psi(x) = x^2$. By the claim above, $|d\psi|(x) < 1$ for all $x \in \Delta$. Using the chain rule, we obtain that $d\varphi = d\psi \circ d\sqrt{\varphi}$. which gives $|d\varphi|(x) = |d\psi|(\sqrt{\varphi}(x))|d\sqrt{\varphi}|(x)$, hence

$$|d\sqrt{\varphi}|(x) = \frac{|d\varphi|(x)|}{|d\psi|(\sqrt{\varphi}(x))} > |d\varphi|(x).$$

■

Riemann mapping theorem

THEOREM: Let $\Omega \subset \Delta$ be a simply connected domain. **Then Ω is biholomorphic to Δ.**

Proof. Step 1: Consider the Kobayashi metric on Ω and Δ, and let $\mathcal{F}$ be the set of all injective holomorphic maps $\Omega \longrightarrow \Delta$. Consider $x \in \Omega$, and let f be a map with $|df|(x)$ maximal in the sense of Kobayashi metric. **Such f exists by Montel's theorem.** Since f lies in the closure of $\mathcal{F}$, and the set of injective maps is closed, **f is injective.**

Step 2: It remains to show that f is surjective. Suppose it is not surjective: $z \notin f(\Omega)$. Taking a composition of f and an isometry of the Poincare disk does not affect $|df|(x)$, hence we may assume that $z = 0$. **Then the function $\sqrt{f}$ is a well defined holomorphic map from Ω to Δ.** By Corollary 1, $|d\sqrt{f}|(x) > |df|(x)$, which is impossible, because it $|df|(x)$ is maximal. ■

Normal families in complete generality

DEFINITION: A set of holomorphic maps $f_\alpha : X \longrightarrow Y$ is called **a normal family** if any sequence $\{f_i\}$ in $\{f_\alpha\}$ has a subsequence converging unformly on compacts.

THEOREM: Let $f_\alpha : X \longrightarrow Y$ be a family of holomorphic maps such that for any point $x \in X$ there exists its neighbourhood with compact closure $K \subset X$ and a Kobayashi hyperbolic open subset $V_K \subset Y$ such that all f_α map K to V_k. Then f_α is a normal family.

EXERCISE: Prove it.

Fatou and Julia sets

DEFINITION: Let $f: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a rational map, and $\{f^i\} = \{f, f \circ f, f \circ f \circ f, ...\}$ the set of all iterations of f. **Fatou set of f** is the set of all points $x \in \mathbb{C}P^1$ such that for some neighbourhood $U \ni x$, the restriction $\{f^i|_U\}$ is a normal family, and **Julia set** is a complement to Fatou set.

EXAMPLE: For the map $f(x) = x^2$, Julia set is the unit circle, and the Fatou set is its complement **(prove it)**.

DEFINITION: **Attractor point** z is a fixed point of f such that $|df|(z) < 1$; the **attractor basin** for z is the set of all $x \in \mathbb{C}P^1$ such that $\lim_i f^i(x) = z$.

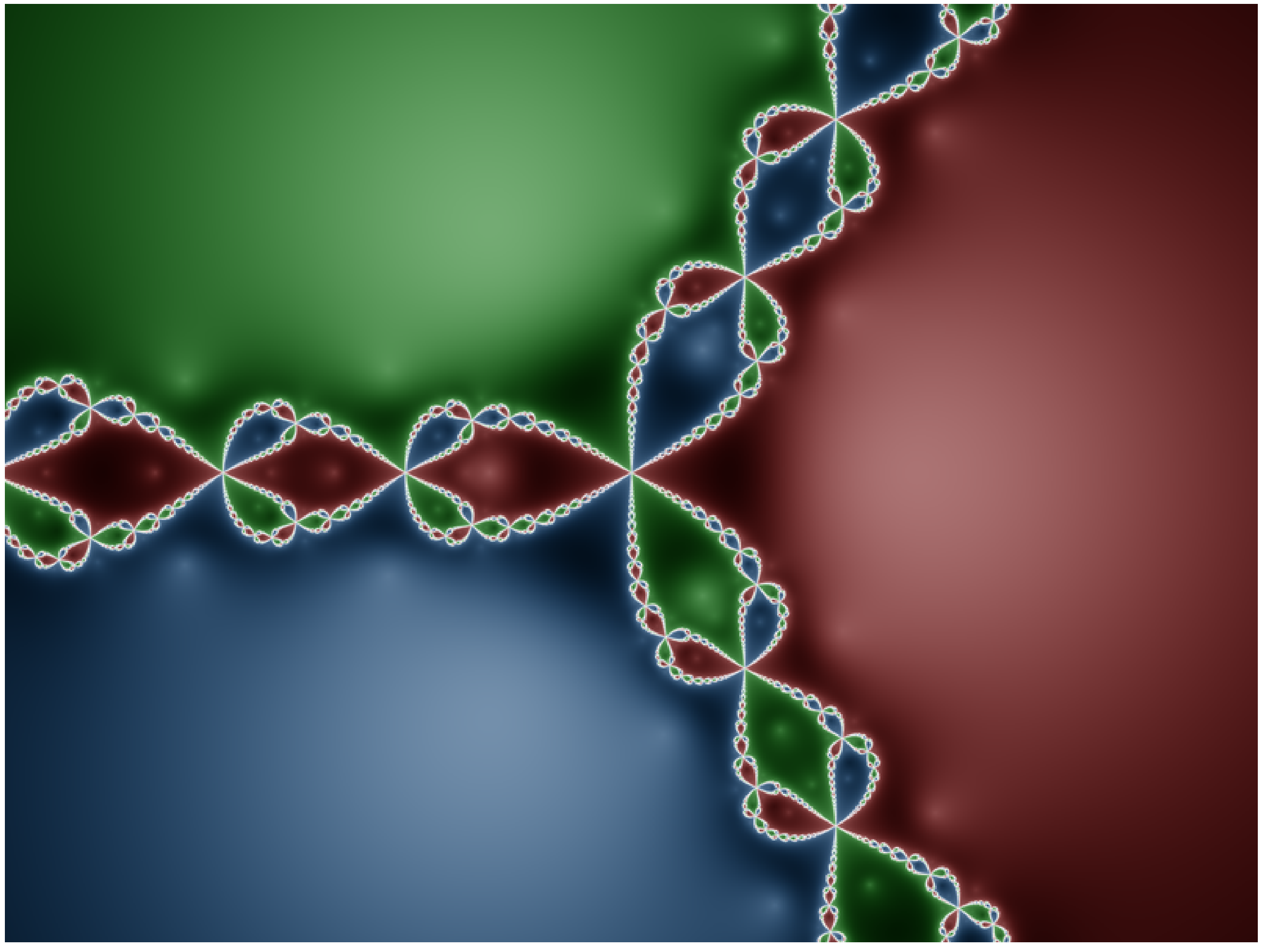
CLAIM: For any fixed point z, **its attractor basin belongs to the Fatou set.**

Proof: Indeed, since $\lim_i f^i(x) = z$ for any point in attractor basin U, $\{f^i\}$ is a normal family on U **(pointwise convergence is equivalent to uniform convergence for bounded holomorphic functions by Arzela-Ascoli theorem).** ■

DEFINITION: **Newton iteration** for solving the polynomial equation $g(z) = 0$: a solution is obtained as a limit $\lim_i f^i(z)$, where $f(z) = z - \frac{g(z)}{g'(z)}$. Indeed, solutions of $g(z) = 0$ are attracting fixed points of f **(check this)**.

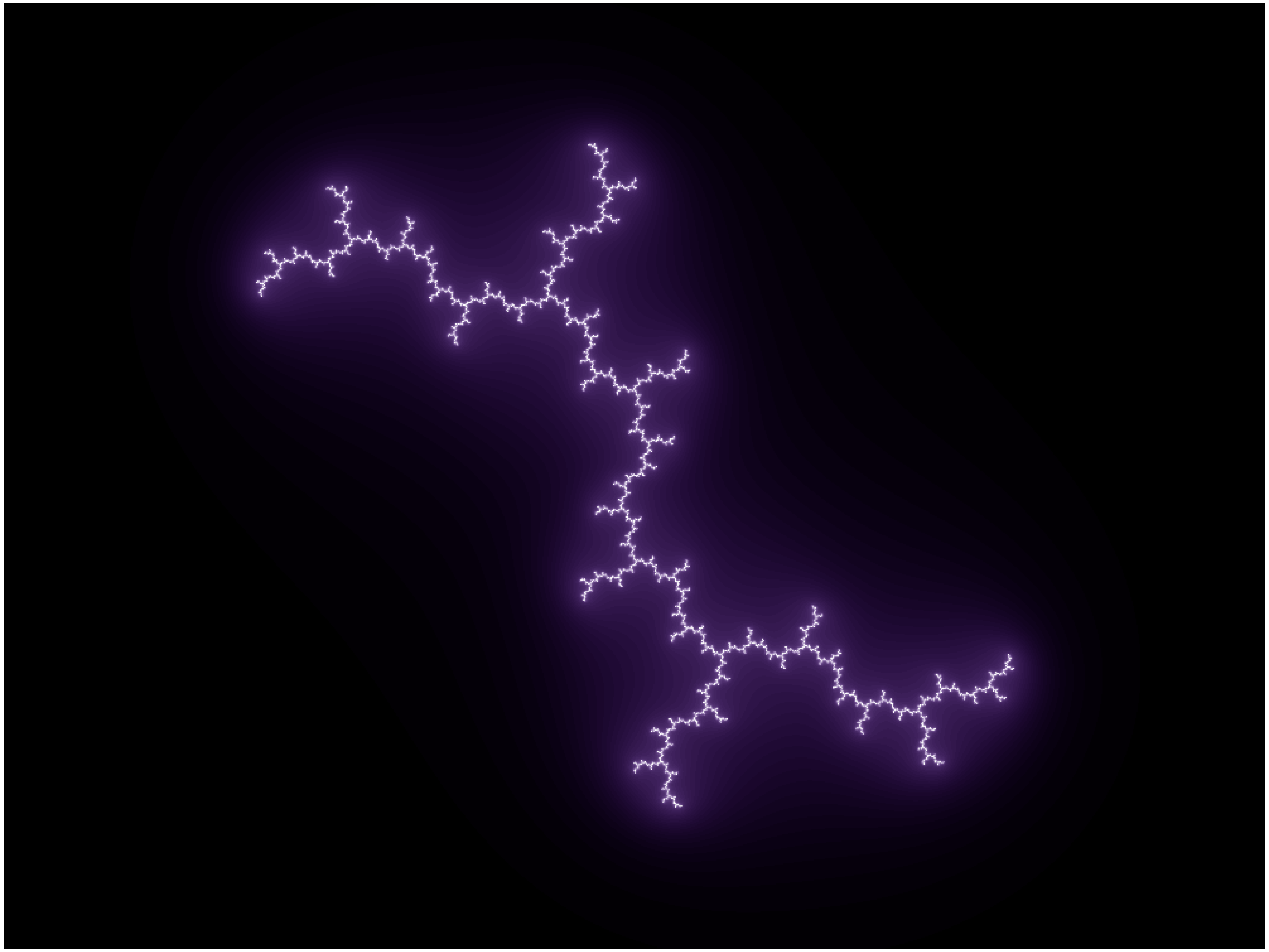
Fatou and Julia sets for $f(z) = \frac{1+2z^3}{3z^2}$

We apply the Newton iteration method to $g(z) = z^3 - 1$.



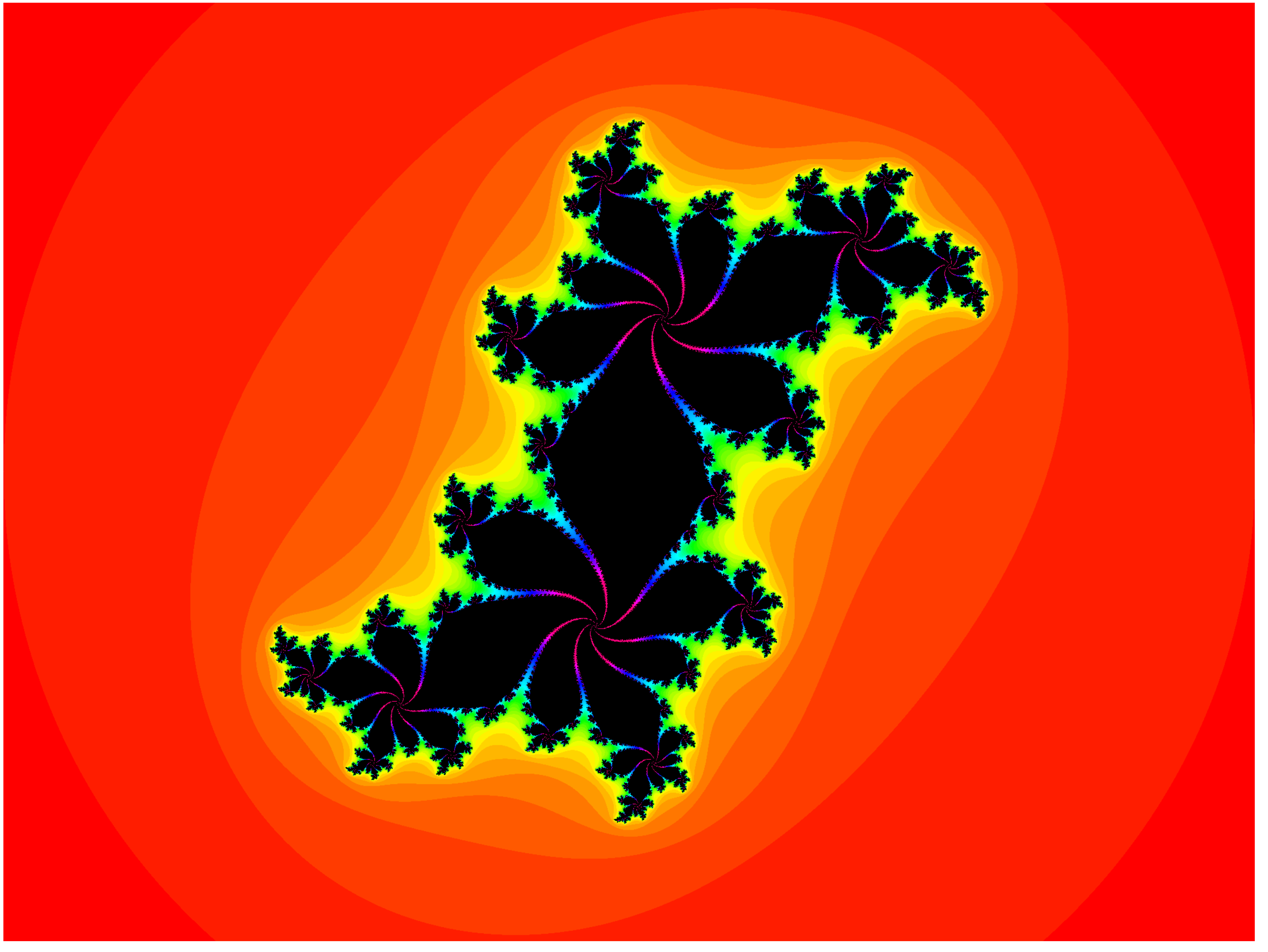
Julia set (in white) for the map $f(z) = \frac{1+2z^3}{3z^2} = z - \frac{g(z)}{g'(z)}$. Coloring of Fatou set according to attractor (the roots of $g(z) = z^3 - 1$).

Julia set for $f(z) = z^2 - \sqrt{-1}$

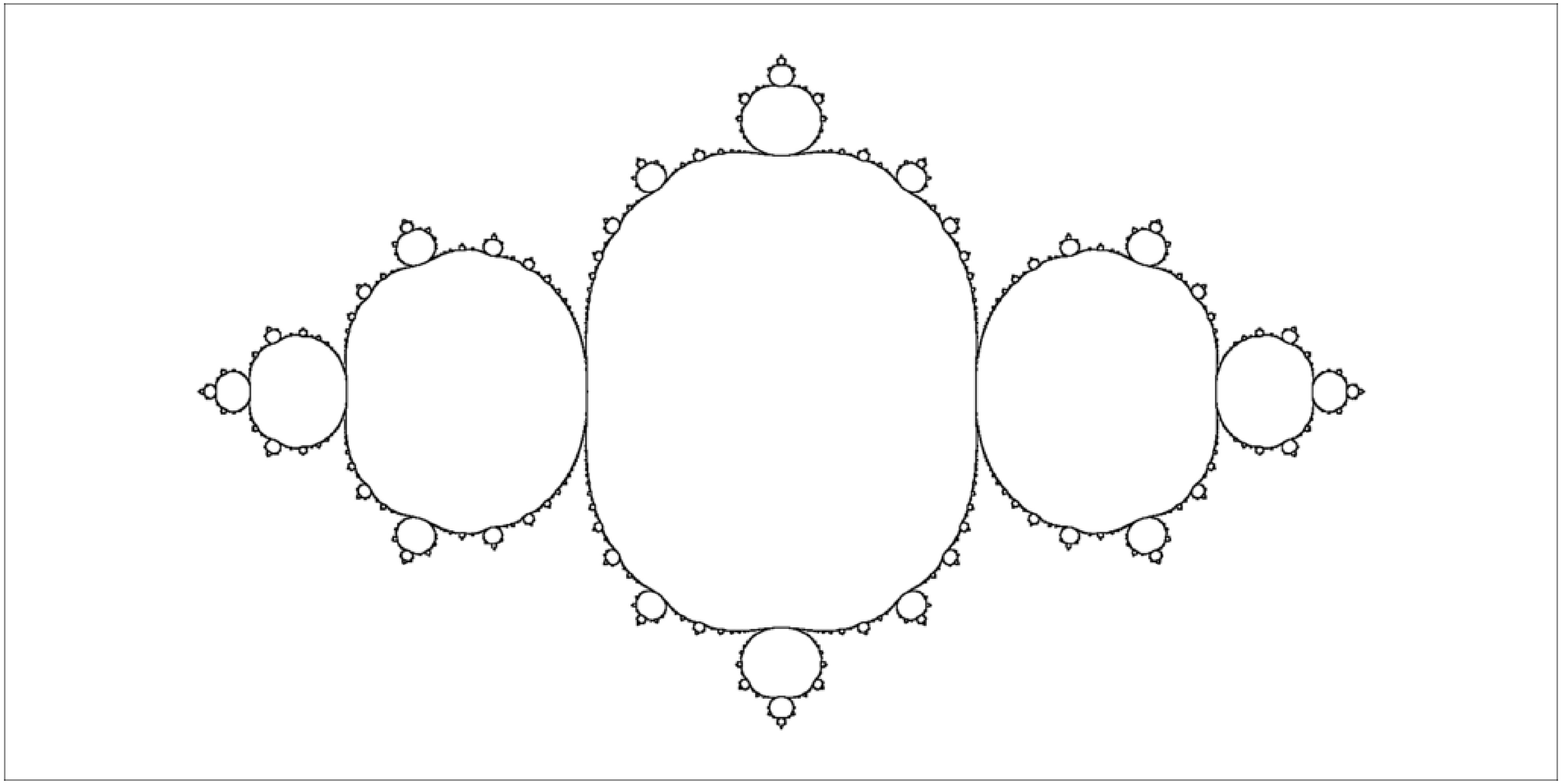


Julia set for $f(z) = z^2 - \sqrt{-1}$ is called **dendrite**.

Julia set for $f(z) = z^2 + 0.12 + 0.6\sqrt{-1}$



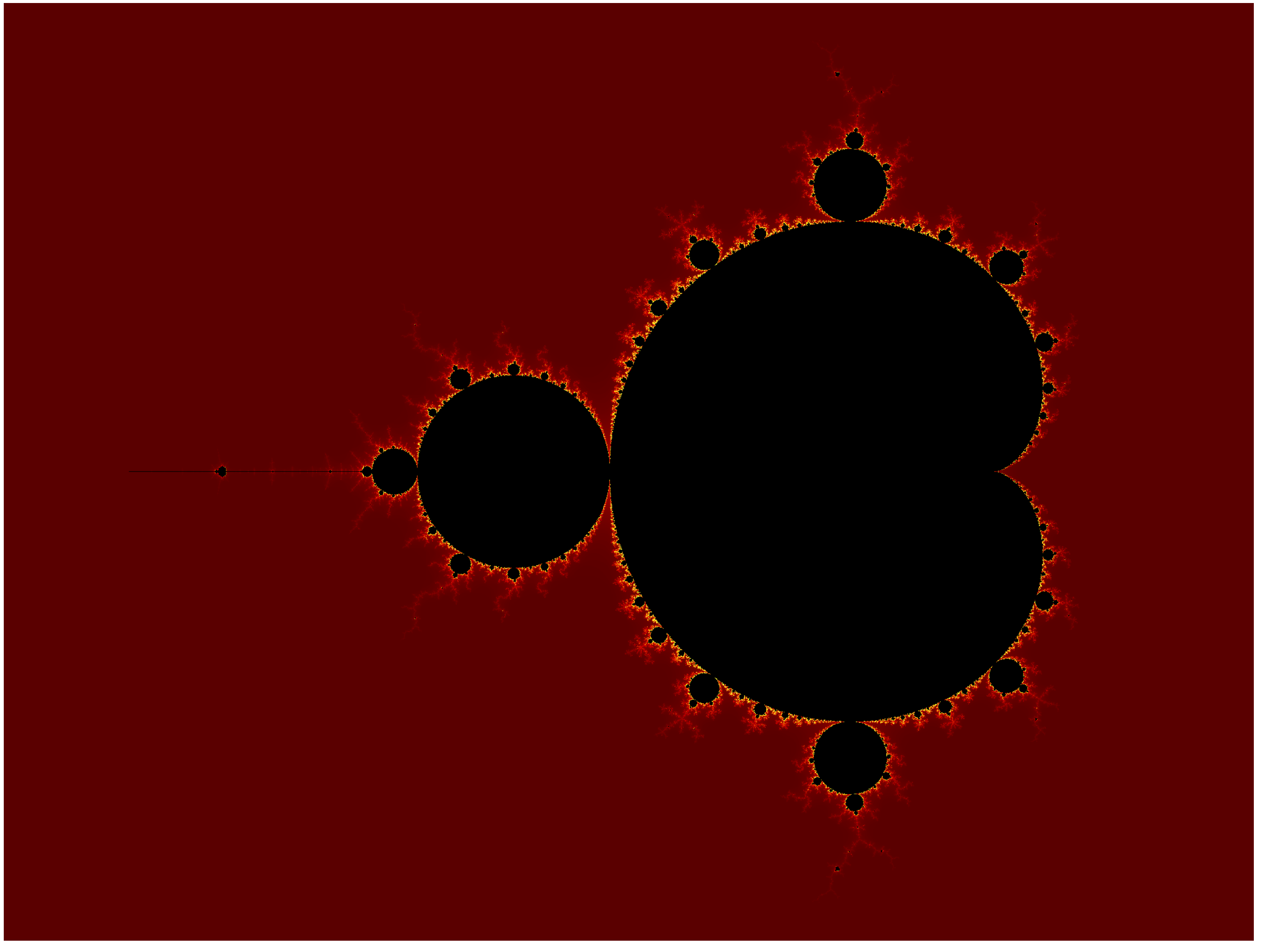
San Marco fractal



San Marco fractal is the Julia set for $f(z) = z^2 - 0.75$

Mandelbrot set

DEFINITION: Mandelbrot set is the set of all c such that 0 belongs to the Fatou set of $f(z) = z^2 + c$.



Properties of Fatou and Julia sets

REMARK: Let $f: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a holomorphic map. **Then the Fatou $F(f)$ and Julia set $J(f)$ of f are f-invariant.**

LEMMA: (Iteration lemma) For each k, $J(f) = J(f^k)$, where f^k is k-th iteration of f.

Proof. Step 1: Clearly, $F(f^k) \subset F(f)$, because $\overline{\{f^k, f^{2k}, f^{3k}, ...\}}$ is compact when $\overline{\{f, f^2, f^3, ...\}}$ is compact.

Proof: Conversely, suppose that $X = F(f^k)$; then $\overline{\{f^k, f^{2k}, f^{3k}, ...\}}$ is compact, but then $\overline{\{f, f^{k+1}, f^{2k+1}, f^{3k+1}, ...\}}$ is also compact as a continuous image image of a compact (the composition is continuous in uniform topology), same for $\overline{\{f^2, f^{k+2}, f^{2k+2}, f^{3k+2}, ...\}}$, and so on. Then $\overline{\{f, f^2, f^3, ...\}}$ is obtained as a union of k compact sets. **Therefore, $F(f) \subset F(f^k)$.** ■

Properties of Fatou and Julia sets (2)

THEOREM: Julia set of polynomial map $f: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ is non-empty, unless $\deg f \leqslant 1$.

Proof: Let $\Delta \subset \mathbb{C}P^1$, and $n(g)$ the number of critical points of a holomorphi function g in Δ. Then $n(g) = \frac{1}{\pi\sqrt{-1}} \int_{\partial\Delta} \frac{g'}{g} dz$, and this number is locally constant in uniform topology if g has no critical points on the boundary. Since the number of critical points of f^i is $i \deg f - 1$, it converges to infinity, hence f^i **cannot converge to a holomorphic function everywhere.** ■