Riemann surfaces: final exam

Rules: Please solve as many of these problems as possible. The final score is computed as $4t+4$, where t is a number of problems solved (2 point problems are counted as 2).

1.1 Almost complex structures and Hodge decomposition

Exercise 1.1. Let ω be a non-degenerate 2-form on a real manifold M. Prove that there exists an almost complex structure I on M and a Hermitian form g such that ω is its Hermitian form, $\omega = g(\cdot, I\cdot)$.

Definition 1.1. Let M be an almost complex manifold, $A: \Lambda^* M \longrightarrow \Lambda^* M$ an endomorphism of the space of differential forms. **Hodge components** of A are operators $A^{p,q}$ such that $A = \sum_{p,q} A^{p,q}$ and $A^{p,q}(\Lambda^{i,j}(M)) \subset \Lambda^{i+p,j+q}(M)$.

Exercise 1.2. Prove that de Rham differential on an almost complex manifold has no more than 4 Hodge components: $d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}$.

1.2 Differential forms on complex manifolds

Exercise 1.3. Let f a real-valued smooth function on a complex manifold which satisfies $dId(f) = 0$. Prove that f is a real part of a holomorphic function or find a counterexample.

Exercise 1.4. Let η be a holomorphic $n-1$-form on a compact complex manifold of complex dimension n. Prove that $d\eta = 0$.

Exercise 1.5. Let f be a real function on a Riemannian surface M such that the top-form $dIdf$ is proportional to the volume form with non-negative coefficient. Prove that that f cannot have a strict maximum anywhere on M.

1.3 Poincare metric

Exercise 1.6 (2 points). Let Δ be a disk in $\mathbb{C}$ and M a complex manifold which is Kobayashi hyperbolic. Prove that any holomorphic map $\Psi: \Delta \longrightarrow M$ can be continuously extended to the boundary of Δ.

Exercise 1.7 (2 points). Let $\Delta^* = \Delta \backslash 0$ be a disk in $\mathbb{C}$ without 0 and M a complex manifold which is Kobayashi hyperbolic. Prove that any holomorphic map $\Psi: \Delta^* \longrightarrow M$ can be continuously extended to 0.