Riemann surfaces: homework assignment 1

Rules: Discussed in class on the next lecture

Exercise 1.1. Let f be a complex analytic function on $\mathbb{C}$ such that there exists $n > 0$ such that $\frac{|f(z)|}{|z|^n}$ is bounded for all z with $|z| > 1$. Prove that f is a polynomial.

Definition 1.1. An **almost complex structure** on a manifold M is a map $I: TM \longrightarrow TM$ such that $I^2 = -\operatorname{Id}$.

Exercise 1.2. Let M be a complex manifold admitting an almost complex structure. Prove that M is oriented.

Exercise 1.3. Prove that any oriented 2-dimensional real manifold admits an almost complex structure, or find a counterexample

Exercise 1.4. Let G be a Lie group (that is, a smooth manifold equipped with a group structure, such that the group operations are given by smooth maps). Prove that $G \times G$ admits an almost complex structure.

Exercise 1.5. Let G be a finite group acting on a manifold M by diffeomorphisms, and M^G the fixed point set of G. Prove that G is a smooth submanifold of M.

Definition 1.2. Let M be an almost complex manifold, and $A: \Lambda^* M \longrightarrow \Lambda^* M$ an endomorphism of the space of differential forms. **Hodge components** of A are operators $A^{p,q}$ such that $A = \sum_{p,q} A^{p,q}$, and $A^{p,q}(\Lambda^{i,j}(M)) \subset \Lambda^{i+p,j+q}(M)$.

Exercise 1.6. Prove that de Rham differential on an almost complex manifold has at most 4 non-zero Hodge components: $d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2}$.