Riemann surfaces

lecture 1

Misha Verbitsky

Université Libre de Bruxelles

October 5, 2016

Complex structure on vector spaces

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $I: V \longrightarrow V$ an automorphism which satisfies $I^2 = -\operatorname{Id}_V$. Such an automorphism is called **a complex structure operator** on V.

We extend the action of I on the tensor spaces $V \otimes V \otimes ... \otimes V \otimes V^* \otimes V^* \otimes ... \otimes V^*$ by multiplicativity: $I(v_1 \otimes ... \otimes w_1 \otimes ... \otimes w_n) = I(v_1) \otimes ... \otimes I(w_1) \otimes ... \otimes I(w_n)$.

Trivial observations:

1. **The eigenvalues α_i of I are $\pm\sqrt{-1}$.** Indeed, $\alpha_i^2 = -1$.

2. **V admits an I-invariant, positive definite scalar product ("metric") g.** Take any metric g_0, and let $g := g_0 + I(g_0)$.

3. **I is orthogonal for such g.**
Indeed, $g(Ix, Iy) = g_0(x, y) + g_0(Ix, Iy) = g(x, y)$.

4. **I diagonalizable over $\mathbb{C}$.** Indeed, any orthogonal matrix is diagonalizable.

5. **There are as many $\sqrt{-1}$-eigenvalues as there are $-\sqrt{-1}$-eigenvalues.**

Comples structure operator in coordinates

This implies that in an appropriate basis in $V \otimes_{\mathbb{R}} \mathbb{C}$, the almost complex structure operator is diagonal, as follows:

$$\left[\begin{array}{cccc|cccc}
\sqrt{-1} & & & & & & & \\
 & \sqrt{-1} & & & & & 0 & \\
 & & \ddots & & & & & \\
 & & & \sqrt{-1} & & & & \\
\hline
 & & & & -\sqrt{-1} & & & \\
 & 0 & & & & -\sqrt{-1} & & \\
 & & & & & & \ddots & \\
 & & & & & & & -\sqrt{-1}
\end{array}\right]$$

We also obtain its normal form in a real basis:

$$\begin{bmatrix}
0 & -1 & & & & & & \\
1 & 0 & & & & & & \\
 & & 0 & -1 & & & & \\
 & & 1 & 0 & & & & \\
 & & & & \ddots & & & \\
 & & & & & \ddots & & \\
 & & & & & & 0 & -1 \\
 & & & & & & 1 & 0
\end{bmatrix}$$

Hermitian structures

DEFINITION: An I-invariant positive definite scalar product on (V, I) is called **an Hermitian metric**, and (V, I, g) – an Hermitian space.

REMARK: Let I be a complex structure operator on a real vector space V, and g – a Hermitian metric. Then **the bilinear form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.** Indeed, $\omega(x, y) = g(x, Iy) = g(Ix, I^2 y) = -g(Ix, y) = -\omega(y, x)$.

DEFINITION: A skew-symmetric form $\omega(x, y)$ is called **an Hermitian form on (V, I)**.

REMARK: In the triple I, g, ω, **each element can recovered from the other two.**

The Grassmann algebra

DEFINITION: Let V be a vector space. Denote by $\Lambda^i V$ the space of antisymmetric polylinear i-forms on V^*, and let $\Lambda^* V := \bigoplus \Lambda^i V$. Denote by $T^{\otimes i} V$ the algebra of **all** polylinear i-forms on V^* ("tensor algebra"), and let $\operatorname{Alt}: T^{\otimes i} V \longrightarrow \Lambda^i V$ be **the antisymmetrization**,

$$\operatorname{Alt}(\eta)(x_1, ..., x_i) := \frac{1}{i!} \sum_{\sigma \in \Sigma_i} (-1)^{\tilde{\sigma}} \eta(x_{\sigma_1}, ..., x_{\sigma_i})$$

where Σ_i is the group of permutations, and $\tilde{\sigma} = 1$ for odd permutations, and 0 for even. Consider the multiplicative operation ("wedge-product") on $\Lambda^* V$, denoted by $\eta \wedge \nu := \operatorname{Alt}(\eta \otimes \nu)$. The space $\Lambda^* V$ with this operation is called **the Grassmann algebra**.

REMARK: **It is an algebra of anti-commutative polynomials.**

Properties of Grassmann algebra:

1. $\dim \Lambda^i V := \binom{\dim V}{i}$, $\dim \Lambda^* V = 2^{\dim V}$.

2. $\Lambda^*(V \oplus W) = \Lambda^*(V) \otimes \Lambda^*(W)$.

The Hodge decomposition in linear algebra

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb{C}} := \Lambda^n_{\mathbb{R}} V \otimes_{\mathbb{R}} C$ admits a decomposition

$$\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q} V$. The resulting decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is called **the Hodge decomposition of the Grassmann algebra**.

Vector fields

DEFINITION: Let X be the vector field on a manifold M, and f a function. Denote by $\operatorname{Lie}_X f$ **the derivatiive** of f along X.

DEFINITION: **A derivation** on a commutative ring is a map $R \xrightarrow{d} R$ satisfying **the Leibniz identity** $d(xy) = d(x)y + xd(y)$.

THEOREM: Each derivation of the ring $C^\infty M$ of smooth functions on M is given by a vector field X; **this correspondence is bijective.**

REMARK: This can be used as a definition of a vector field.

EXERCISE: Prove that **a commutator of two derivations is again a derivation.**

REMARK: Vector fields are the same as derivations of $C^\infty M$. This allows us to define **the commutator of two vector fields** as the commutator of the corresponding derivations.

DEFINITION: Denote by TM the bundle of vector fields, and by $\Lambda^1 M$ or T^* the dual bundle, called **the bundle of 1-forms**. For any $f \in C^\infty M$, the operation $X \longrightarrow \operatorname{Lie}_X f$ is linear as a function of X, hence it defines a section of T^*M. We denote this section df, and call it **the differential** of f.

De Rham algebra

DEFINITION: Let $\Lambda^* M$ denote the vector bundle with the fiber $\Lambda^* T_x^* M$ at $x \in M$ ($\Lambda^* T^* M$ is the Grassman algebra of the cotangent space $T_x^* M$). The sections of $\Lambda^i M$ are called **differential i-forms**. The algebraic operation "wedge product" defined on differential forms is $C^\infty M$-linear; the space $\Lambda^* M$ of all differential forms is called **the de Rham algebra**.

REMARK: $\Lambda^0 M = C^\infty M$.

THEOREM: **There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} ...$ satisfying the following properties**

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta \in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

Stokes' theorem: Let η be $n-1$-form on n-manifold M with a boundary ∂M. **Then $\int_M d\eta = \int_{\partial M} \eta$.**

Holomorphic functions

DEFINITION: Let $I: TM \longrightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\operatorname{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

EXAMPLE: $M = \mathbb{C}^n$, with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$, and $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$.

DEFINITION: A function $f: M \longrightarrow \mathbb{C}$ on an almost complex manifold is called **holomorphic** if $df \in \Lambda^{1,0}(M)$.

REMARK: For some almost complex manifolds, **there are no holomorphic functions at all**, even locally.

Holomorphic functions on $\mathbb{C}^n$

THEOREM: Let $f: M \longrightarrow \mathbb{C}$ be a differentiable function on an open subset $M \subset \mathbb{C}^n$, with the natural almost complex structure. **Then the following are equivalent.**

(1) f **is holomorphic**.

(2) The differential $df: TM \longrightarrow \mathbb{C}$, considered as a form on the vector space $T_xM = T_x\mathbb{C}^n = \mathbb{C}^n$ **is $\mathbb{C}$-linear.**

(3) For any complex affine line $L \in \mathbb{C}^n$, the restriction $f|_L = \mathbb{C}$ is **holomorphic (complex analytic) as a function of one complex variable.**

(4) f is expressed as a sum of Taylor series around any point $(z_1, ..., z_n) \in M$:

$$f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}.$$

(here we assume that the complex numbers t_i satisfy $|t_i| < \varepsilon$, where ε depends on f and M).

Proof: (1) and (2) are tautologically equivalent. Equivalence of (1) and (3) is also clear, because a restriction of $\theta \in \Lambda^{1,0}(M)$ to a line is a $(1,0)$-form on a line, and, conversely, if df is of type (1,0) on each complex line, it is of type (1,0) on TM, which is implied by the following linear-algebraic observation.

Holomorphic functions on $\mathbb{C}^n$ (2)

THEOREM: Let $f: M \longrightarrow \mathbb{C}$ be a differentiable function on an open subset $M \subset \mathbb{C}^n$, with the natural almost complex structure. **Then the following are equivalent.**
(1) f **is holomorphic**.
(2) The differential $df: TM \longrightarrow \mathbb{C}$, considered as a form on the vector space $T_xM = T_x\mathbb{C}^n = \mathbb{C}^n$ **is $\mathbb{C}$-linear.**
(3) For any complex affine line $L \in \mathbb{C}^n$, the restriction $f|_L = \mathbb{C}$ is **holomorphic (complex analytic) as a function of one complex variable.**
(4) f is expressed as a sum of Taylor series around any point $(z_1, ..., z_n) \in M$.

LEMMA: Let $\eta \in V^* \otimes \mathbb{C}$ be a complex-valued linear form on a vector space (V, I) equipped with a complex structure. **Then $\eta \in \Lambda^{1,0}(V)$ if and only if its restriction to any I-invariant 2-dimensional subspace L belongs to $\Lambda^{1,0}(L)$.**

EXERCISE: Prove it.

(4) clearly implies (2). It remains to show that (1) implies (4).

Taylor decomposition from Cauchy formula

Taylor series decomposition on a line is implied by the Cauchy formula:

$$\int_{\partial\Delta} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1}\, f(a),$$

where $\Delta \subset \mathbb{C}$ is a disk, $a \in \Delta$ any point, and z coordinate on $\mathbb{C}$. Indeed, in this case,

$$2\pi\sqrt{-1}\, f(a) = \sum_{i\geqslant 0} a^i \int_{\partial\Delta} f(z)(z^{-1})^{i+1},$$

because $\frac{1}{z-a} = z^{-1}\sum_{i\geqslant 0}(az^{-1})^i$.

Cauchy formula

Let's prove Cauchy formula, using Stokes' theorem. Since the space $\Lambda^{1,0}\mathbb{C}$ is 1-dimensional, $df \wedge dz = 0$ for any holomorphic function on $\mathbb{C}$. This gives

CLAIM: A function on a disk $\Delta \subset \mathbb{C}$ **is holomorphic if and only if the form $\eta := f dz$ is closed** (that is, satisfies $d\eta = 0$). ■

Now, let S_ε be a radius ε circle around a point $a \in \Delta$, Δ_ε its interior, and $\Delta_0 := \Delta \backslash \Delta_\varepsilon$. Stokes' theorem gives

$$0 = \int_{\Delta_0} d\left(\frac{f(z)dz}{z-a}\right) = -\int_{S_\varepsilon} \frac{f(z)dz}{z-a} + \int_{\partial\Delta} \frac{f(z)dz}{z-a},$$

hence Cauchy formula would follow if we show that $\lim_{\varepsilon\to 0} \int_{S_\varepsilon} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1} f(a)$.

Assuming for simplicity $a = 0$ and parametrizing the circle S_ε by $\varepsilon e^{\sqrt{-1}\,t}$, we obtain

$$\int_{S_\varepsilon} \frac{f(z)dz}{z} = \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}\,t})}{\varepsilon e^{\sqrt{-1}\,t}} d(\varepsilon e^{\sqrt{-1}\,t}) =$$

$$= \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}\,t})}{\varepsilon e^{\sqrt{-1}\,t}} \sqrt{-1}\,\varepsilon e^{\sqrt{-1}\,t} dt = \int_0^{2\pi} f(\varepsilon e^{\sqrt{-1}\,t})\sqrt{-1}\,dt$$

as ε tends to 0, $f(\varepsilon e^{\sqrt{-1}\,t})$ tends to $f(0)$, and this integral goes to $2\pi\sqrt{-1} f(0)$.

Sheaves

DEFINITION: A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

DEFINITION: A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and presheaves: examples

Examples of sheaves:

* Space of continuous functions

* Space of smooth functions, any differentiability class

* Space of real analytic functions

Examples of presheaves which are not sheaves:

* Space of constant functions **(why?)**

* Space of bounded functions **(why?)**

Ringed spaces

A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \xrightarrow{\Psi} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \xrightarrow{\Psi} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $\psi^* f := f \circ \Psi$ belongs to the ring $\mathcal{F}\Big(\Psi^{-1}(U)\Big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

EXAMPLE: Let M be a manifold of class C^i and let $C^i(U)$ be the space of functions of this class. **Then C^i is a sheaf of functions, and (M, C^i) is a ringed space.**

REMARK: Let $f: X \longrightarrow Y$ be a smooth map of smooth manifolds. Since a pullback $f^*\mu$ of a smooth function $\mu \in C^\infty(M)$ is smooth, **a smooth map of smooth manifolds defines a morphism of ringed spaces.**

Complex manifolds

DEFINITION: A holomorphic function on $\mathbb{C}^n$ is a function $f: \mathbb{C}^n \longrightarrow \mathbb{C}$ such that df is complex linear, that is $df \in \Lambda^{1,0}(M)$.

REMARK: Holomorphic functions form a sheaf.

DEFINITION: A complex manifold M is a ringed space which is locally isomorphic to an open ball in $\mathbb{C}^n$ with a sheaf of holomorphic functions.

REMARK: In other words, M **is covered with open balls embedded to** $\mathbb{C}^n$ and transition functions (being coordinate functions for one ball considered in other coordinate system) **are holomorphic.**

Complex manifolds and almost complex manifolds

DEFINITION: Standard almost complex structure is $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$ on $\mathbb{C}^n$ with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$.

DEFINITION: A map $\Psi : (M, I) \longrightarrow (N, J)$ from an almost complex manifold to an almost complex manifold is called **holomorphic** if $\Psi^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M)$.

REMARK: This is the same as $d\Psi$ being complex linear; for standard almost complex structures, **this is the same as the coordinate components of Ψ being holomorphic functions.**

DEFINITION: A complex manifold is a manifold equipped with an atlas with charts identified with open subsets of $\mathbb{C}^n$ and transition functions holomorphic.

Integrability of almost complex structures

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, and $\mathcal{O}_M$ is determined by I as explained above. ■

Frobenius form

CLAIM: Let $B \subset TM$ be a sub-bundle of a tangent bundle of a smooth manifold. Given vector fiels $X, Y \in B$, consider their commutator $[X, Y]$, and lets $\Psi(X, Y) \in TM/B$ be the projection of $[X, Y]$ to TM/B. **Then $\Psi(X, Y)$ is $C^\infty(M)$-linear in X, Y:**

$$\Psi(fX, Y) = \Psi(X, fY) = f\Psi(X, Y).$$

Proof: Leibnitz identity gives $[X, fY] = f[X, Y] + X(f)Y$, and the second term belongs to B, hence does not influence the projection to TM/B. ■

DEFINITION: This form is called **the Frobenius form** of the sub-bundle $B \subset TM$. This bundle is called **involutive**, or **integrable**, or **holonomic** if $\Psi = 0$.

EXERCISE: Give an example of a non-integrable sub-bundle.

Formal integrability

DEFINITION: An almost complex structure I on (M, I) is called **formally integrable** if $[T^{1,0}M, T^{1,0}] \subset T^{1,0}$, that is, if $T^{1,0}M$ is involutive.

DEFINITION: The Frobenius form $\Psi \in \Lambda^{2,0}M \otimes TM$ is called **the Nijenhuis tensor**.

CLAIM: If a complex structure I on M is integrable, it is formally integrable.

Proof: Locally, the bundle $T^{1,0}(M)$ is generated by d/dz_i, where z_i are complex coordinates. These vector fields commute, hence satisfy $[d/dz_i, d/dz_j] \in T^{1,0}(M)$. This means that the Frobenius form vanishes. ∎

THEOREM: (Newlander-Nirenberg)
A complex structure I on M is integrable if and only if it is formally integrable.

Proof: (real analytic case) next lecture.

REMARK: In dimension 1, formal integrability is automatic. Indeed, $T^{1,0}M$ is 1-dimensional, hence all skew-symmetric 2-forms on $T^{1,0}M$ vanish.