Riemann surfaces

lecture 3: Riemannian structures

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Almost complex manifolds (reminder)

DEFINITION: Let $I : TM \longrightarrow TM$ be an endomorphism of a tangent bundle satisfying $I^2 = -\operatorname{Id}$. Then I is called **almost complex structure operator**, and the pair (M, I) **an almost complex manifold**.

EXAMPLE: $M = \mathbb{C}^n$, with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$, and $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$.

DEFINITION: Let (V, I) be a space equipped with a complex structure $I : V \longrightarrow V$, $I^2 = -\operatorname{Id}$. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

DEFINITION: A function $f : M \longrightarrow \mathbb{C}$ on an almost complex manifold is called **holomorphic** if $df \in \Lambda^{1,0}(M)$.

REMARK: For some almost complex manifolds, **there are no holomorphic functions at all**, even locally.

Complex manifolds and almost complex manifolds (reminder)

DEFINITION: **Standard almost complex structure** is $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$ on $\mathbb{C}^n$ with complex coordinates $z_i = x_i + \sqrt{-1}\, y_i$.

DEFINITION: A map $\Psi: (M,I) \longrightarrow (N,J)$ from an almost complex manifold to an almost complex manifold is called **holomorphic** if $\Psi^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M)$.

REMARK: This is the same as $d\Psi$ being complex linear; for standard almost complex structures, **this is the same as the coordinate components of Ψ being holomorphic functions.**

DEFINITION: **A complex manifold** is a manifold equipped with an atlas with charts identified with open subsets of $\mathbb{C}^n$ and transition functions holomorphic.

Integrability of almost complex structures (reminder)

DEFINITION: An almost complex structure I on a manifold is called **integrable** if any point of M has a neighbourhood U diffeomorphic to an open subset of $\mathbb{C}^n$, in such a way that the almost complex structure I is induced by the standard one on $U \subset \mathbb{C}^n$.

CLAIM: Complex structure on a manifold M uniquely determines an integrable almost complex structure, and is determined by it.

Proof: Complex structure on a manifold M is determined by the sheaf of holomorphic functions $\mathcal{O}_M$, and $\mathcal{O}_M$ is determined by I as explained above. ■

Frobenius form (reminder)

CLAIM: Let $B \subset TM$ be a sub-bundle of a tangent bundle of a smooth manifold. Given vector fiels $X, Y \in B$, consider their commutator $[X, Y]$, and lets $\Psi(X, Y) \in TM/B$ be the projection of $[X, Y]$ to TM/B. **Then $\Psi(X, Y)$ is $C^\infty(M)$-linear in X, Y:**

$$\Psi(fX, Y) = \Psi(X, fY) = f\Psi(X, Y).$$

Proof: Leibnitz identity gives $[X, fY] = f[X, Y] + X(f)Y$, and the second term belongs to B, hence does not influence the projection to TM/B. ■

DEFINITION: This form is called **the Frobenius form** of the sub-bundle $B \subset TM$. This bundle is called **involutive**, or **integrable**, or **holonomic** if $\Psi = 0$.

EXERCISE: Give an example of a non-integrable sub-bundle.

Formal integrability (reminder)

DEFINITION: An almost complex structure I on (M, I) is called **formally integrable** if $[T^{1,0}M, T^{1,0}] \subset T^{1,0}$, that is, if $T^{1,0}M$ is involutive.

DEFINITION: The Frobenius form $\Psi \in \Lambda^{2,0}M \otimes TM$ is called **the Nijenhuis tensor**.

CLAIM: If a complex structure I on M is integrable, it is formally integrable.

Proof: Locally, the bundle $T^{1,0}(M)$ is generated by d/dz_i, where z_i are complex coordinates. These vector fields commute, hence satisfy $[d/dz_i, d/dz_j] \in T^{1,0}(M)$. This means that the Frobenius form vanishes. ∎

THEOREM: (Newlander-Nirenberg)
A complex structure I on M is integrable if and only if it is formally integrable.

REMARK: In dimension 1, formal integrability is automatic. Indeed, $T^{1,0}M$ is 1-dimensional, hence all skew-symmetric 2-forms on $T^{1,0}M$ vanish.

Riemannian manifolds

DEFINITION: Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: For any $x, y \in M$, and any path $\gamma: [a,b] \longrightarrow M$ connecting x and y, consider **the length** of γ defined as $L(\gamma) = \int_\gamma |\frac{d\gamma}{dt}| dt$, where $|\frac{d\gamma}{dt}| = h(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})^{1/2}$. Define **the geodesic distance** as $d(x,y) = \inf_\gamma L(\gamma)$, where infimum is taken for all paths connecting x and y.

EXERCISE: Prove that the **geodesic distance satisfies triangle inequality and defines metric on M.**

EXERCISE: Prove that **this metric induces the standard topology on M.**

EXAMPLE: Let $M = \mathbb{R}^n$, $h = \sum_i dx_i^2$. **Prove that the geodesic distance coincides with $d(x,y) = |x-y|$.**

EXERCISE: Using partition of unity, **prove that any manifold admits a Riemannian structure.**

Hermitian structures

DEFINITION: A Riemannia metric h on an almost complex manifold is called **Hermitian** if $h(x,y) = h(Ix, Iy)$.

REMARK: Given any Riemannian metric g on an almost complex manifold, **a Hermitian metric h can be obtained as $h = g + I(g)$, where $I(g)(x,y) = g(I(x), I(y))$.**

REMARK: Let I be a complex structure operator on a real vector space V, and g – a Hermitian metric. Then **the bilinear form $\omega(x,y) := g(x, Iy)$ is skew-symmetric.** Indeed, $\omega(x,y) = g(x, Iy) = g(Ix, I^2y) = -g(Ix, y) = -\omega(y,x)$.

DEFINITION: A skew-symmetric form $\omega(x,y)$ is called **an Hermitian form on (V,I)**.

REMARK: In the triple I, g, ω, each element can recovered from the other two.

Conformal structure

DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

DEFINITION: Conformal structure on M is a class of conformal equivalence of Riemannian metrics.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**. Conversely, any metric conformally equivalent to Hermitian is Hermitian.

REMARK: The last statement is clear from the definition, and true in any dimension.

To prove that any two Hermitian metrics are conformally equivalent, **we need to consider the standard $U(1)$-action on a complex vector space** (see the next slide).

Standard $U(1)$-action

DEFINITION: Let (V, I) be a real vector space equipped with a complex structure, $U(1)$ the group of unit complex numbers, $U(1) = e^{\sqrt{-1}\,\pi t}$, $t \in \mathbb{R}$. Define the action of $U(1)$ on V as follows: $\rho(t) = e^{tI}$. This is called **the standard $U(1)$-action on a complex vector space**. To prove that this formula defines an action if $U(1) = \mathbb{R}/2\pi\mathbb{Z}$, it suffices to show that $e^{2\pi I} = 1$, which is clear from the eigenvalue decomposition of I.

CLAIM: Let (V, I, h) be a Hermitian vector space, and $\rho: U(1) \longrightarrow GL(V)$ the standard $U(1)$-action. **Then h is $U(1)$-invariant.**

Proof: It suffices to show that $\frac{d}{dt}(h(\rho(t)x, \rho(t)x) = 0$. However, $\frac{d}{dt}e^{tI}(x)\Big|_{t=t_0} = I(e^{t_0 I}(x))$, hence

$$\frac{d}{dt}(h(\rho(t)x, \rho(t)x) = h(I(\rho(t)x), \rho(t)x) + h(\rho(t)x, I(\rho(t)x)) = 2\omega(x, x) = 0.$$

■

Hermitian metrics in $\dim_{\mathbb{R}} = 2$.

COROLLARY: Let h, h' be Hermitian metrics on a space (V, I) of real dimension 2. **Then h and h' are proportional.**

Proof: h and h' are constant on any $U(1)$-orbit. Multiplying h' by a constant, we may assume that $h = h'$ on a $U(1)$-orbit $U(1)x$. Then $h = h'$ everywhere, because **for each non-zero vector $v \in V$, $tv \in U(1)x$ for some $t \in \mathbb{R}$, giving $h(v, v) = t^{-2}h(tv, tv) = t^{-2}h'(tv, tv) = h'(v, v)$.** ■

DEFINITION: Given two Hermitian forms h, h' on (V, I), with $\dim_{\mathbb{R}} V = 2$, we denote by $\frac{h'}{h}$ a constant t such that $h' = th$.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**.

Proof: $h' = \frac{h'}{h} h$. ■

EXERCISE: Prove that **Riemannian structure on M is uniquely defined by its conformal class and its Riemannian volume form**.

Conformal structures and almost complex structures

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I, let ν be the conformal class of its Hermitian metric (it is unique as shown above). **Then ν determines I uniquely.**

Proof: Choose a Riemannian structure h compatible with the conformal structure ν. Since M is oriented, the group $SO(2) = U(1)$ acts in its tangent bundle in a natural way: $\rho: U(1) \longrightarrow GL(TM)$. Rescaling h does not change this action, hence it is determined by ν. Now, define I as $\rho(\sqrt{-1})$; then $I^2 = \rho(-1) = -\operatorname{Id}$. Since $U(1)$ acts by isometries, this almost complex structure is compatible with h and with ν. ■

DEFINITION: A Riemann surface is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure.

EXERCISE: Prove that **a continuous map from one Riemannian surface to another is holomorphic if and only if it preserves the conformal structure** almost everywhere.

Homogeneous spaces

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\mathrm{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has** $M = G/H$, where $H = \mathrm{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\mathrm{St}_x(G)^g = \mathrm{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\mathrm{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\mathrm{St}_x(G)$ on T_xM.

DEFINITION: A tensor Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant tensor on $\mathrm{St}_x(G)$. For any $y \in M$ obtained as $y = g(x)$, consider the tensor Φ_y on T_yM obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \mathrm{St}_x(G)$. Since Φ is h-invariant, **the tensor Φ_y is independent from the choice of g.**

We proved

THEOREM: **Homogeneous tensors on $M = G/H$ are in bijective correspondence with isotropy invariant tensors on T_xM,** for any $x \in M$.

■

Space forms

DEFINITION: Simply connected space form is a homogeneous Riemannian manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

The Riemannian metric is defined in the next slide.

Riemannian metric on space forms

LEMMA: Let $G = SO(n)$ act on $\mathbb{R}^n$ in a natural way. **Then there exists a unique G-invariant symmetric 2-form:** the standard Euclidean metric.

Proof: Let g, g' be two G-invariant symmetric 2-forms. Since S^{n-1} is an orbit of G, we have $g(x,x) = g(y,y)$ for any $x, y \in S^{n-1}$. Multiplying g' by a constant, we may assume that $g(x,x) = g'(x,x)$ for any $x \in S^{n-1}$. **Then $g(\lambda x, \lambda x) = g'(\lambda x, \lambda x)$ for any $x \in S^{n-1}$, $\lambda \in \mathbb{R}$;** however, all vectors can be written as λx. ■

COROLLARY: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Proof: The isotropy group is $SO(n-1)$ in all three cases, and the previous lemma can be applied. ■