Riemann surfaces

lecture 4: Möbius group

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Hermitian and conformal structures (reminder)

DEFINITION: Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: A Riemannia metric h on an almost complex manifold is called **Hermitian** if $h(x,y) = h(Ix, Iy)$.

DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

DEFINITION: **Conformal structure** on M is a class of conformal equivalence of Riemannian metrics.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**. Conversely, any metric conformally equivalent to Hermitian is Hermitian.

Conformal structures and almost complex structures (reminder)

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I, let ν be the conformal class of its Hermitian metric (it is unique as shown above). **Then ν determines I uniquely.**

Proof: Choose a Riemannian structure h compatible with the conformal structure ν. Since M is oriented, the group $SO(2) = U(1)$ acts in its tangent bundle in a natural way: $\rho: U(1) \longrightarrow GL(TM)$. Rescaling h does not change this action, hence it is determined by ν. Now, define I as $\rho(\sqrt{-1}\,)$; then $I^2 = \rho(-1) = -\operatorname{Id}$. Since $U(1)$ acts by isometries, this almost complex structure is compatible with h and with ν. ■

DEFINITION: **A Riemann surface** is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure.

Homogeneous spaces (reminder)

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\operatorname{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \operatorname{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\operatorname{St}_x(G)^g = \operatorname{St}_y(G)$: **all the isotropy groups are conjugate.**

Space forms (reminder)

DEFINITION: Simply connected space form is a homogeneous Riemannian manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

The Riemannian metric is defined by the following lemma.

LEMMA: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Poincaré-Koebe uniformization theorem

DEFINITION: A **Riemannian manifold of constant curvature** is a Riemannian manifold which is locally isometric to a space form.

THEOREM: (Poincaré-Koebe uniformization theorem) Let M be a Riemann surface. **Then M admits a unique complete metric of constant curvature in the same conformal class.**

COROLLARY: Any Riemann surface is a quotient of a space form X by a discrete group of isometries $\Gamma \subset \operatorname{Iso}(X)$.

COROLLARY: Any simply connected Riemann surface is conformally equivalent to a space form.

REMARK: We shall prove some cases of the uniformization theorem in later lectures.

Today's subject: classify conformal automorphisms of all space forms.

Some low-dimensional Lie group isomorphisms

DEFINITION: Lie algebra of a Lie group G is the Lie algebra Lie(G) of left-invariant vector fields. **Adjoint representation** of G is the standard action of G on Lie(G). For a Lie group $G = GL(n)$, $SL(n)$, etc., $PGL(n)$, $PSL(n)$, etc. denote the image of G in GL(Lie(G)) with respect to the adjoint action.

REMARK: This is the same as a quotient G/Z by the centre of G.

DEFINITION: Define $SO(1,2)$ as the group of orthogonal matrices on a 3-dimensional space equipped with a scalar product of signature (1,2), and $U(1,1)$ as the group of complex linear maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ preserving a pseudio-Hermitian form of signature (1,1).

THEOREM: The groups $PU(1,1)$, $PSL(2,\mathbb{R})$ and $SO(1,2)$ are isomorphic.

Proof: Isomorphism $PU(1,1) = SO(1,2)$ will be established later in this lecture. To see $PSL(2,\mathbb{R}) \cong SO(1,2)$, consider **the Killing form** κ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, $a, b \longrightarrow \operatorname{Tr}(ab)$. **Check that it has signature $(1,2)$. Then the image of $SL(2,\mathbb{R})$ in automorphisms of its Lie algebra is mapped to $SO(\mathfrak{sl}(2,\mathbb{R}),\kappa) = SO(1,2)$.** Both groups are 3-dimensional, hence it is an isomorphism. ■

Laurent power series

THEOREM: (Laurent theorem)
Let f be a holomorphic function on an annulus (that is, a ring)

$$R = \{z \mid \alpha < |z| < \beta\}.$$

Then f can be expressed as a Laurent power series $f(z) = \sum_{i\in\mathbb{Z}} z^i a_i$ converging in R.

Proof: Same as Cauchy formula. ■

REMARK: This theorem remains valid if $\alpha = 0$ and $\beta = \infty$.

REMARK: A function $\varphi : \mathbb{C}^* \longrightarrow \mathbb{C}$ uniquely determines its Laurent power series. Indeed, residue of $z^k\varphi$ in 0 is $\sqrt{-1}\, 2\pi a_{-k-1}$.

REMARK: Let $\varphi : \mathbb{C}^* \longrightarrow \mathbb{C}$ be a holomorphic function, and $\varphi = \sum_{i\in\mathbb{Z}} z^i a_i$ its Laurent power series. **Then $\psi(z) := \varphi(z^{-1})$ has Laurent polynomial $\psi = \sum_{i\in\mathbb{Z}} z^{-i} a_i$.**

Affine coordinates on $\mathbb{C}P^1$

DEFINITION: We identify $\mathbb{C}P^1$ with the set of pairs $x : y$ defined up to equivalence $x : y \sim \lambda x : \lambda y$, for each $\lambda \in \mathbb{C}^*$. This representation is called **homogeneous coordinates**. **Affine coordinates** are $1 : z$ for $x \neq 0$, $z = y/x$ and $z : 1$ for $y \neq 0$, $z = x/y$. The corresponding gluing functions are given by the map $z \longrightarrow z^{-1}$.

DEFINITION: **Meromorphic function** is a quotient f/g, where f, g are holomorphic and $g \neq 0$.

REMARK: A holomorphic map $\mathbb{C} \longrightarrow \mathbb{C}P^1$ is the same as a pair of maps $f : g$ up to equivalence $f : g \sim fh : gh$. **In other words, holomorphic maps $\mathbb{C} \longrightarrow \mathbb{C}P^1$ are identified with meromorphic functions on $\mathbb{C}$.**

REMARK: In homogeneous coordinates, an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{C})$ acts as $x : y \longrightarrow ax + by : cx + dy$. Therefore, in affine coordinates it acts as $z \longrightarrow \frac{az+b}{cz+d}$.

Möbius transforms

DEFINITION: Möbius transform is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: The group $PGL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphially.

The following theorem will be proven later in this lecture.

THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group of Möbius transforms is an isomorphism.

Claim 1: Let $\varphi: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a holomorphic automorphism, $\varphi_0: \mathbb{C} \longrightarrow \mathbb{C}P^1$ its restriction to the chart $z:1$, and $\varphi_\infty: \mathbb{C} \longrightarrow \mathbb{C}P^1$ its restriction $1:z$. We consider φ_0, φ_∞ as meromorphic functions on $\mathbb{C}$. **Then $\varphi_\infty = \varphi_0(z^{-1})^{-1}$.**

Möbius transforms and $PGL(2,\mathbb{C})$

THEOREM: The natural map from $PGL(2,\mathbb{C})$ to the group $\operatorname{Aut}(\mathbb{C}P^1)$ of Möbius transforms is an isomorphism.

Proof. Step 1: Let $\varphi \in \operatorname{Aut}(\mathbb{C}P^1)$. Since $PSL(2,\mathbb{C})$ acts transitively on pairs of points $x \neq y$ in $\mathbb{C}P^1$, by composing φ with an appropriate element in $PGL(2,\mathbb{C})$ we can assume that $\varphi(0) = 0$ and $\varphi(\infty = \infty$. This means that we may consider the restrictions φ_0 and φ_∞ of φ to the affine charts as a holomorphic functions on these charts, $\varphi_0, \varphi_\infty : \mathbb{C} \longrightarrow \mathbb{C}$.

Step 2: Let $\varphi_0 = \sum_{i>0} a_i z^i$, $a_1 \neq 0$. Claim 1 gives

$$\varphi_\infty(z) = \varphi_0(z^{-1})^{-1} = a_1 z (1 + \sum_{i\geqslant 2} \frac{a_i}{a_1} z^{-i})^{-1}.$$

Unless $a_i = 0$ for all $i \geqslant 2$, this Laurent series has singularities in 0 and cannot be holomorphic. **Therefore φ_0 is a linear function**, and it belongs to $PGL(2,\mathbb{C})$. ■

Lemma 1: Let φ be a Möbius transform fixing $\infty \in \mathbb{C}P^1$. **Then $\varphi(z) = az + b$ for some $a, b \in \mathbb{C}$ and all $z = z : 1 \in \mathbb{C}P^1$.**

Proof: Let $A \in PGL(2,\mathbb{C})$ be a map acting on $\mathbb{C} = \mathbb{C}P^1 \backslash \infty$ as parallel transport mapping $\varphi(0)$ to 0. Then $\varphi \circ A$ is a Moebius transform which fixes ∞ and 0. As shown in Step 2 above, it is a linear function. ■

Properties of Möbius transform

DEFINITION: A circle in S^2 is an orbit of a 1-parametric isometric rotation subgroup $U \subset PGL(2,\mathbb{C})$.

PROPOSITION: The action of $PGL(2,\mathbb{C})$ on $\mathbb{C}P^1$ maps circles to circles.

Proof. Step 1: Consider a pseudo-Hermitian form h on $V = \mathbb{C}^2$ of signature (1,1). Let h_+ be a positive definite Hermitian form on V. There exists a basis $x, y \in V$ such that $h_+ = \sqrt{-1}\, x \otimes \overline{x} + \sqrt{-1}\, y \otimes \overline{y}$ (that is, x, y is orthonormal with respect to h_+) and $h = -\sqrt{-1}\, \alpha x \otimes \overline{x} + \sqrt{-1}\, \beta y \otimes \overline{y}$, with $\alpha > 0$, $\beta < 0$ real numbers. Then $\{z \mid h(z,z) = 0\}$ is invariant under the rotation $x, y \longrightarrow x, e^{\sqrt{-1}\,\theta} y$, hence **it is a circle.**

Step 2: Clearly, all circles are obtained this way.

Step 3: $PGL(2,\mathbb{C})$ maps pseudo-Hermitian forms to pseudo-Hermitian forms of the same signature, and therefore **preserves circles.** ■

Orbits of compact one-parametric subgroups in $PSL(2,\mathbb{C})$

LEMMA: Let $G \cong S^1$ be a compact one-parametric subgroup in $PSL(2,\mathbb{C})$. **Then any G-orbit in $\mathbb{C}P^1$ is a circle.**

Proof. Step 1: Let $V = \mathbb{C}^2$, and consider the natural projection map $\pi : SL(V) \longrightarrow PSL(2,\mathbb{C}) = SL(V)/\pm 1$. Then $\tilde{G} = \pi^{-1}(G)$ is compact. Choose a $\tilde{G}$-invariant Hermitian metric h_1 on V, and let h be the standard Hermitiann metric. Since $GL(2,\mathbb{C})$ acts on the set of Hermitian metrics transitively **(prove it)**, there exists $u \in GL(V)$ such that $u(h) = h_1$. By definition, circles on $\mathbb{C}P^1$ are orbits of one-parametric subgroups in $U(V,h)$. **Since $u(\tilde{G})$ is a one-parametric subgroup in $U(V,h)$, its orbit is a circle.**

Step 2: From Step 1, we obtain that any orbit of G is u^{-1}(*circle*). Since u^{-1} is a Moebius transform, and Moebius transforms preserve circles, this orbit is a circle. ■

Conformal automorphisms of $\mathbb{C}$

THEOREM: (Riemann removable singularity theorem) Let $f: \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function which is holomorphic outside of a finite set. **Then f is holomorphic.**

Proof: Use the Cauchy formula. ■

THEOREM: All conformal automorphisms of $\mathbb{C}$ can be expressed by $z \longrightarrow az + b$, where a, b are complex numbers, $a \neq 0$.

Proof: Let φ be a conformal automorphism of $\mathbb{C}$. The Riemann removable singularity theorem implies that **φ can be extended to a holomorphic automorphism of $\mathbb{C}P^1$.** Indeed, $\mathbb{C}P^1$ is obtained as a 1-point compactification of $\mathbb{C}$, and any continuous map from $\mathbb{C}$ to $\mathbb{C}$ is extended to a continuous map on $\mathbb{C}P^1$. Now, Lemma 1 implies that $\varphi(z) = az + b$. ■