Riemann surfaces

lecture 5: automorphisms of Poincaré plane

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Hermitian and conformal structures (reminder)

DEFINITION: Let $h \in \operatorname{Sym}^2 T^*M$ be a symmetric 2-form on a manifold which satisfies $h(x,x) > 0$ for any non-zero tangent vector x. Then h is called **Riemannian metric**, of **Riemannian structure**, and (M,h) **Riemannian manifold**.

DEFINITION: A Riemannia metric h on an almost complex manifold is called **Hermitian** if $h(x,y) = h(Ix, Iy)$.

DEFINITION: Let h, h' be Riemannian structures on M. These Riemannian structures are called **conformally equivalent** if $h' = fh$, where f is a positive smooth function.

DEFINITION: **Conformal structure** on M is a class of conformal equivalence of Riemannian metrics.

CLAIM: Let I be an almost complex structure on a 2-dimensional Riemannian manifold, and h, h' two Hermitian metrics. **Then h and h' are conformally equivalent**. Conversely, any metric conformally equivalent to Hermitian is Hermitian.

Conformal structures and almost complex structures (reminder)

REMARK: The following theorem implies that almost complex structures on a 2-dimensional oriented manifold are equivalent to conformal structures.

THEOREM: Let M be a 2-dimensional oriented manifold. Given a complex structure I, let ν be the conformal class of its Hermitian metric (it is unique as shown above). **Then ν determines I uniquely.**

Proof: Choose a Riemannian structure h compatible with the conformal structure ν. Since M is oriented, the group $SO(2) = U(1)$ acts in its tangent bundle in a natural way: $\rho: U(1) \longrightarrow GL(TM)$. Rescaling h does not change this action, hence it is determined by ν. Now, define I as $\rho(\sqrt{-1}\,)$; then $I^2 = \rho(-1) = -\operatorname{Id}$. Since $U(1)$ acts by isometries, this almost complex structure is compatible with h and with ν. ■

DEFINITION: **A Riemann surface** is a complex manifold of dimension 1, or (equivalently) an oriented 2-manifold equipped with a conformal structure.

Homogeneous spaces (reminder)

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\operatorname{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has** $M = G/H$, where $H = \operatorname{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\operatorname{St}_x(G)^g = \operatorname{St}_y(G)$: **all the isotropy groups are conjugate.**

Space forms (reminder)

DEFINITION: Simply connected space form is a homogeneous Riemannian manifold of one of the following types:

positive curvature: S^n (an n-dimensional sphere), equipped with an action of the group $SO(n+1)$ of rotations

zero curvature: $\mathbb{R}^n$ (an n-dimensional Euclidean space), equipped with an action of isometries

negative curvature: $SO(1,n)/SO(n)$, equipped with the natural $SO(1,n)$-action. This space is also called **hyperbolic space**, and in dimension 2 **hyperbolic plane** or **Poincaré plane** or **Bolyai-Lobachevsky plane**

The Riemannian metric is defined by the following lemma.

LEMMA: Let $M = G/H$ be a simply connected space form. **Then M admits a unique, up to a constant multiplier, G-invariant Riemannian form.**

Some low-dimensional Lie group isomorphisms *reminder)

DEFINITION: Lie algebra of a Lie group G is the Lie algebra Lie(G) of left-invariant vector fields. **Adjoint representation** of G is the standard action of G on Lie(G). For a Lie group $G = GL(n)$, $SL(n)$, etc., $PGL(n)$, $PSL(n)$, etc. denote the image of G in $GL(\text{Lie}(G))$ with respect to the adjoint action.

REMARK: This is the same as a quotient G/Z by the centre of G.

DEFINITION: Define $SO(1,2)$ as the group of orthogonal matrices on a 3-dimensional space equipped with a scalar product of signature (1,2), and $U(1,1)$ as the group of complex linear maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ preserving a pseudio-Hermitian form of signature (1,1).

THEOREM: The groups $PU(1,1)$, $PSL(2,\mathbb{R})$ and $SO(1,2)$ are isomorphic.

Proof: Isomorphism $PU(1,1) = SO(1,2)$ will be established later in this lecture. To see $PSL(2,\mathbb{R}) \cong SO(1,2)$, consider **the Killing form** κ on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, $a,b \longrightarrow \text{Tr}(ab)$. **Check that it has signature $(1,2)$. Then the image of $SL(2,\mathbb{R})$ in automorphisms of its Lie algebra is mapped to $SO(\mathfrak{sl}(2,\mathbb{R}),\kappa) = SO(1,2)$.** Both groups are 3-dimensional, hence it is an isomorphism. ■

Möbius transforms (reminder)

DEFINITION: **Möbius transform** is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: **The group $PGL(2,\mathbb{C})$ acts on $\mathbb{C}P^1$ holomorphially.**

THEOREM: **The natural map from $PGL(2,\mathbb{C})$ to the group of Möbius transforms is an isomorphism**.

DEFINITION: **A circle in S^2** is an orbit of a 1-parametric isometric rotation subgroup $U \subset PGL(2,\mathbb{C})$.

PROPOSITION: **The action of $PGL(2,\mathbb{C})$ on $\mathbb{C}P^1$ maps circles to circles.**

THEOREM: **All conformal automorphisms of $\mathbb{C}$ can be expressed by $z \longrightarrow az+b$,** where a, b are complex numbers, $a \neq 0$.

Schwartz lemma

CLAIM: (maximum principle) Let f be a holomorphic function defined on an open set U. **Then f cannot have strict maxima in U. If f has non-strict maxima, it is constant.**

EXERCISE: Prove the maximum principle.

LEMMA: (Schwartz lemma) Let $f : \Delta \longrightarrow \Delta$ be a map from disk to itself fixing 0. **Then $|f'(0)| \leqslant 1$, and equality can be realized only if $f(z) = \alpha z$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.**

Proof: Consider the function $\varphi := \frac{f(z)}{z}$. Since $f(0) = 0$, it is holomorphic, and since $f(\Delta) \subset \Delta$, on the boundary $\partial\Delta$ we have $|\varphi||_{\partial\Delta} \leqslant 1$. Now, **the maximum principle implies that $|f'(0)| = |\varphi(0)| \leqslant 1$,** and equality is realized only if $\varphi = const$. ■

Conformal automorphisms of the disk act transitively

CLAIM: Let $\Delta \subset \mathbb{C}$ be the unit disk. **Then the group** $\operatorname{Aut}(\Delta)$ **of its holomorphic automorphisms acts on** Δ **transitively.**

Proof. Step 1: Let $V_a(z) = \frac{z-a}{1-\overline{a}z}$ for some $a \in \Delta$. Then $V_a(0) = -a$. To prove transitivity, it remains to show that $V_a(\Delta) = \Delta$.

Step 2: For $|z| = 1$, we have

$$|V_a(z)| = |V_a(z)||z| = \left|\frac{z\overline{z} - a\overline{z}}{1 - \overline{a}z}\right| = \left|\frac{1 - a\overline{z}}{1 - \overline{a}z}\right| = 1.$$

Therefore, V_a preserves the circle. Maximum principle implies that V_a maps its interior to its interior.

Step 3: To prove invertibility, we interpret V_a as an element of $PGL(2, \mathbb{C})$. ■

Transitive action is determined by a stabilizer of a point

Lemma 2: Let $M = G/H$ be a homogeneous space, and $\Psi: G_1 \longrightarrow G$ a homomorphism such that G_1 acts on M transitively and $\operatorname{St}_x(G_1) = \operatorname{St}_x(G)$. **Then $G_1 = G$.**

Proof: Since any element in $\ker\Psi$ belongs to $\operatorname{St}_x(G_1) = \operatorname{St}_x(G) \subset G$, the homomorphism Ψ is injective. It remais only to show that Ψ is surjective.

Let $g \in G$. Since G_1 acts on M transitively, $gg_1(x) = x$ for some $g_1 \in G_1$. Then $gg_1 \in \operatorname{St}_x(G_1) = \operatorname{St}_x(G) \subset \operatorname{im} G_1$. This gives $g \in G_1$. ■

Group of conformal automorphisms of the disk

REMARK: The group $PU(1,1) \subset PGL(2,\mathbb{C})$ of unitary matrices preserving a pseudo-Hermitian form h of signature (1,1) acts on a disk $\{l \in \mathbb{C}P^1 \ | \ h(l,l) > 0\}$ by holomorphic automorphisms.

COROLLARY: Let $\Delta \subset \mathbb{C}$ be the unit disk, $\operatorname{Aut}(\Delta)$ the group of its conformal automorphisms, and $\Psi: PU(1,1) \longrightarrow \operatorname{Aut}(\Delta)$ the map constructed above. **Then Ψ is an isomorphism.**

Proof: We use Lemma 2. Both groups act on Δ transitively, hence **it suffices only to check that $\operatorname{St}_x(PU(1,1)) = S^1$ and $\operatorname{St}_x(\operatorname{Aut}(\Delta)) = S^1$.** The first isomorphism is clear, because the space of unitary automorphisms fixing a vector v is $U(v^\perp)$. The second isomorphism follows from Schwartz lemma **(prove it!)**. ■

COROLLARY: Let h be a homogeneous metric on $\Delta = PU(1,1)/S^1$. **Then (Δ, h) is conformally equivalent to $(\Delta,$ flat metric$)$.**

Proof: The group $\operatorname{Aut}(\Delta) = PU(1,1)$ acts on Δ holomorphically, that is, preserving the conformal structure of the flat metric. However, homogeneous conformal structure on $PU(1,1)/S^1$ is unique for the same reason the homogeneous metric is unique up to a contant multiplier **(prove it)**. ■

Upper half-plane

REMARK: The map $z \longrightarrow -\sqrt{-1}\,(z-1)^{-1}$ induces a diffeomorphism from the unit disc in $\mathbb{C}$ to the upper half-plane $\mathbb{H}$ **(prove it)**.

PROPOSITION: The group $\operatorname{Aut}(\Delta)$ **acts on the upper half-plane** $\mathbb{H}$ **as** $z \xrightarrow{A} \frac{az+b}{cz+d}$, **where** $a, b, c, d \in \mathbb{R}$, **and** $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$.

REMARK: The group of such A is naturally identified with $PSL(2,\mathbb{R}) \subset PSL(2,\mathbb{C})$.

Proof: The group $PSL(2,\mathbb{R})$ preserves the line $\operatorname{im} z = 0$, hence acts on $\mathbb{H}$ by conformal automorphisms. The stabilizer of a point is S^1 **(prove it).** Now, Lemma 2 implies that $PSL(2,\mathbb{R}) = PU(1,1)$. ■

REMARK: We have shown that $\mathbb{H} = SO(1,2)/S^1$, hence **$\mathbb{H}$ is conformally equivalent to the hyperbolic space.**

Upper half-plane as a Riemannian manifold

DEFINITION: Poincaré half-plane is the upper half-plane equipped with a homogeneous metric of constant negative curvature constructed above.

THEOREM: Let (x, y) be the usual coordinates on the upper half-plane $\mathbb{H}$. **Then the Riemannian structure s on $\mathbb{H}$ is written as $s = const\frac{dx^2+dy^2}{y^2}$.**

Proof: Since the complex structure on $\mathbb{H}$ is the standard one and all Hermitian structures are proportional, we obtain that $s = \mu(dx^2+dy^2)$, where $\mu \in C^\infty(\mathbb{H})$. **It remains to find μ, using the fact that s is $PSL(2, \mathbb{R})$-invariant.**

For each $a \in \mathbb{R}$, the parallel transport $x \longrightarrow x + a$ fixes s, hence μ is a function of y. For any $\lambda \in \mathbb{R}^{>0}$, the map $H_\lambda(x) = \lambda x$ also fixes s; since $\mathbb{H}_\lambda(dx^2+dy^2) = \lambda^2 dx^2 + dy^2$, we have $\mu(\lambda x) = \lambda^{-2}\mu(x)$. ∎

Geodesics on Riemannian manifold

DEFINITION: Minimising geodesic in a Riemannian manifold is a piecewise smooth path connecting x to y such that its length is equal to the geodesic distance. **Geodesic** is a piecewise smooth path γ such that for any $x \in \gamma$ there exists a neighbourhood of x in γ which is a minimising geodesic.

EXERCISE: Prove that a big circle in a sphere is a geodesic. Prove that an interval of a big circle of length $\leqslant \pi$ is a minimising geodesic.

Geodesics in Poincaré half-plane

THEOREM: Geodesics on a Poincaré half-plane are vertical straight lines and their images under the action of $SL(2,\mathbb{R})$.

Proof. Step 1: Let $a, b \in \mathbb{H}$ be two points satisfying $\operatorname{Re} a = \operatorname{Re} b$, and l the line connecting these two points. Denote by Π the orthogonal projection from $\mathbb{H}$ to the vertical line connecting a to b. For any tangent vector $v \in T_z\mathbb{H}$, one has $|D\pi(v)| \leqslant |v|$, and the equality means that v is vertical (prove it). Therefore, **a projection of a path γ connecting a to b to l has length $\leqslant L(\gamma)$, and the equality is realized only if γ is a straight vertical interval.**

Step 2: For any points a, b in the Poincaré half-plane, **there exists an isometry mapping (a, b) to a pair of points (a_1, b_1) such that $\operatorname{Re}(a_1) = \operatorname{Re}(b_1)$. (Prove it!)**

Step 3: Using Step 2, we prove that **any geodesic γ on a Poincaré half-plane is obtained as an isometric image of a straight vertical line:** $\gamma = v(\gamma_0)$, $v \in \operatorname{Iso}(\mathbb{H}) = PSL(2,\mathbb{R})$ ■

Geodesics in Poincaré half-plane

CLAIM: Let S be a circle or a straight line on a complex plane $\mathbb{C} = \mathbb{R}^2$, and S_1 closure of its image in $\mathbb{C}P^1$ inder the natural map $z \longrightarrow 1 : z$. **Then S_1 is a circle, and any circle in $\mathbb{C}P^1$ is obtained this way.**

Proof: The circle $S_r(p)$ of radius r centered in $p \in \mathbb{C}$ is given by equation $|p - z| = r$, in homogeneous coordinates it is $|px - z|^2 = r|x|^2$. This is the zero set of the pseudo-Hermitian form $h(x, z) = |px - z|^2 - |x|^2$, hence it is a circle. ■

COROLLARY: Geodesics on the Poincaré half-plane are vertical straight lines and half-circles orthogonal to the line $\operatorname{im} z = 0$ in the intersection points.

Proof: We have shown that geodesics in the Poincaré half-plane are Möbius transforms of straight lines orthogonal to $\operatorname{im} z = 0$. However, any Möbius transform preserves angles and maps circles or straight lines to circles or straight lines. ■