Riemann surfaces

lecture 6: Kobayashi metrics

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Möbius transforms (reminder)

DEFINITION: Möbius transform is a conformal (that is, holomorphic) diffeomorphism of $\mathbb{C}P^1$.

REMARK: The group $PGL(2,\mathbb{C})$ **acts on** $\mathbb{C}P^1$ **holomorphially.**

THEOREM: The natural map from $PGL(2,\mathbb{C})$ **to the group** $\mathsf{Aut}(\mathbb{C}P^1)$ **of Möbius transforms is an isomorphism**.

DEFINITION: A circle in S^2 is an orbit of a 1-parametric isometric rotation subgroup $U \subset PGL(2,\mathbb{C})$.

PROPOSITION: The action of $PGL(2,\mathbb{C})$ **on** $\mathbb{C}P^1$ **maps circles to circles.**

Proof: Consider a pseudo-Hermitian form h on $V = \mathbb{C}^2$ of signature (1,1). Let h_+ be a positive definite Hermitian form on V. There exists a basis $x, y \in V$ such that $h_+ = \sqrt{-1}\, x \otimes \bar{x} + \sqrt{-1}\, y \otimes \bar{y}$ (that is, x, y is orthonormal with respect to h_+) and $h = -\sqrt{-1}\, \alpha x \otimes \bar{x} + \sqrt{-1}\, \beta y \otimes \bar{y}$, with $\alpha > 0$, $\beta < 0$ real numbers. Then $\{z \mid h(z, z) = 0\}$ is invariant under the rotation $x, y \longrightarrow x, e^{\sqrt{-1}\,\theta} y$, hence **it is a circle.** ∎

Conformal automorphisms of the disk (reminder)

CLAIM: Let $\Delta \subset \mathbb{C}$ be the unit disk. **Then the group** $\mathrm{Aut}(\Delta)$ **of its holomorphic automorphisms acts on** Δ **transitively.**

Proof: Let $V_a(z) = \frac{z-a}{1-\bar{a}z}$ for some $a \in \Delta$. Then $V_a(0) = -a$. To prove transitivity, it remains to show that $V_a(\Delta) = \Delta$, which is implied from

$$|V_a(z)| = |V_a(z)||z| = \left|\frac{z\bar{z} - a\bar{z}}{1 - \bar{a}z}\right| = \left|\frac{1 - a\bar{z}}{1 - \bar{a}z}\right| = 1.$$

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REMARK: The group $PU(1,1) \subset PGL(2,\mathbb{C})$ of unitary matrices preserving a pseudo-Hermitian form h of signature (1,1) acts on a disk $\{l \in \mathbb{C}P^1 \mid h(l,l) > 0\}$ by holomorphic automorphisms.

COROLLARY: Let $\Delta \subset \mathbb{C}$ be the unit disk, $\mathrm{Aut}(\Delta)$ the group of its conformal automorphisms, and $\Psi : PU(1,1) \longrightarrow \mathrm{Aut}(\Delta)$ the map constructed above. **Then Ψ is an isomorphism.**

COROLLARY: Let h be a homogeneous metric on $\Delta = PU(1,1)/S^1$. **Then (Δ, h) is conformally equivalent to $(\Delta, \text{flat metric})$.**

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Upper half-plane (reminder)

REMARK: The map $z \longrightarrow -\sqrt{-1}/2 - (z - \sqrt{-1})^{-1}$ induces a diffeomorphism from the unit disc in $\mathbb{C}$ to the upper half-plane $\mathbb{H}$.

PROPOSITION: The group $\mathsf{Aut}(\Delta)$ **acts on the upper half-plane** $\mathbb{H}$ **as** $z \xrightarrow{A} \frac{az+b}{cz+d}$**, where** $a, b, c, d \in \mathbb{R}$**, and** $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0$**.**

REMARK: The group of such A is naturally identified with $PSL(2, \mathbb{R}) \subset PSL(2, \mathbb{C})$.

REMARK: We have shown that $\mathbb{H} = SO(1,2)/S^1$, hence $\mathbb{H}$ **is conformally equivalent to the hyperbolic space.**

Upper half-plane as a Riemannian manifold (reminder)

DEFINITION: Poincaré half-plane is the upper half-plane equipped with a homogeneous metric of constant negative curvature constructed above.

THEOREM: Let (x, y) be the usual coordinates on the upper half-plane $\mathbb{H}$. **Then the Riemannian structure s on $\mathbb{H}$ is written as** $s = const\frac{dx^2 + dy^2}{y^2}$.

Proof: Since the complex structure on $\mathbb{H}$ is the standard one and all Hermitian structures are proportional, we obtain that $s = \mu(dx^2 + dy^2)$, where $\mu \in C^\infty(\mathbb{H})$. **It remains to find μ, using the fact that s is $PSL(2, \mathbb{R})$-invariant.**

For each $a \in \mathbb{R}$, the parallel transport $x \longrightarrow x + a$ fixes s, hence μ is a function of y. For any $\lambda \in \mathbb{R}^{>0}$, the map $H_\lambda(x) = \lambda x$ also fixes s; since $\mathbb{H}_\lambda(dx^2 + dy^2) = \lambda^2 dx^2 + dy^2$, we have $\mu(\lambda x) = \lambda^{-2}\mu(x)$. ∎

Geodesics on Riemannian manifold (reminder)

THEOREM: **Geodesics on a Poincaré half-plane are vertical straight lines and their images under the action of $SL(2, \mathbb{R})$.**

CLAIM: Let S be a circle or a straight line on a complex plane $\mathbb{C} = \mathbb{R}^2$, and S_1 closure of its image in $\mathbb{C}P^1$ inder the natural map $z \longrightarrow 1 : z$. **Then S_1 is a circle, and any circle in $\mathbb{C}P^1$ is obtained this way.**

Proof: The circle $S_r(p)$ of radius r centered in $p \in \mathbb{C}$ is given by equation $|p - z| = r$, in homogeneous coordinates it is $|px - z|^2 = r|x|^2$. This is the zero set of the pseudo-Hermitian form $h(x, z) = |px - z|^2 - |x|^2$, hence it is a circle. ∎

COROLLARY: Geodesics on the Poincaré half-plane are vertical straight lines and half-circles orthogonal to the line $\operatorname{im} z = 0$ in the intersection points.

Proof: We have shown that geodesics in the Poincaré half-plane are Möbius transforms of straight lines orthogonal to $\operatorname{im} z = 0$. However, any Möbius transform preserves angles and maps circles or straight lines to circles or straight lines. ∎

Poincaré metric on disk

DEFINITION: Poincaré metric on a unit disk $\Delta \subset \mathbb{C}$ is an $\mathrm{Aut}(\Delta)$-invariant metric **(it is unique up to a constant multiplier)**.

DEFINITION: Let $f : \ M \longrightarrow M_1$ be a map of metric spaces. Then f is called C-**Lipschitz** if $d(x, y) \geqslant Cd(f(x), f(y))$. A map is called **Lipschitz** if it is C-Lipschitz for some $C > 0$.

THEOREM: (Schwartz-Pick lemma)
Any holomorphic map $\varphi : \ \Delta \longrightarrow \Delta$ **from a unit disk to itself is 1-Lipschitz with respect to Poicaré metric.**

Proof. Step 1: We need to prove that for each $x \in \Delta$ the norm of the differential satisfies $|D\varphi_x| \leqslant 1$. Since the automorphism group acts on Δ transitively by isometries, **it suffices to prove that** $|D\varphi_x| \leqslant 1$ **when** $x = 0$ **and** $\varphi(x) = 0$**.**

Step 2: This is Schwartz lemma. ∎

Kobayashi pseudometric

DEFINITION: Pseudometric on M is a function $d : M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric: $d(x, y) = d(y, x)$ and satisfies the triangle inequality $d(x, y) + d(y, z) \geqslant d(x, z)$.

REMARK: Let $\mathfrak{D}$ be a set of pseudometrics. **Then** $d_{\max}(x, y) := \sup_{d \in \mathfrak{D}} d(x, y)$ **is also a pseudometric.**

DEFINITION: The **Kobayashi pseudometric** on a complex manifold M is $d_{\max}$ for the set $\mathfrak{D}$ of all pseudometrics such that any holomorphic map from the Poincaré disk to M is distance-decreasing.

EXERCISE: Prove that **the distance between points x, y in Kobayashi pseudometric is infimum of the Poincaré distance over all sets of Poincaré disks connecting x to y.**

EXAMPLE: The Kobayashi pseudometric on $\mathbb{C}$ vanishes.

CLAIM: Any holomorphic map $X \xrightarrow{\varphi} Y$ is 1-Lipschitz with respect to the Kobayashi pseudometric.

Proof: If $x \in X$ is connected to x' by a sequence of Poincare disks $\Delta_1, ..., \Delta_n$, then $\varphi(x)$ is connected to $\varphi(x')$ by $\varphi(\Delta_1), ..., \varphi(\Delta_n)$. $\blacksquare$

Kobayashi hyperbolic manifolds

COROLLARY: Let $B \subset \mathbb{C}^n$ be a unit ball, and $x, y \in B$ points with coordinates $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$. Since x_i, y_i belongs to Δ, it makes sense to compute the Poincare distance $d_P(x_i, y_i)$. **Then** $d_K(x, y) \geqslant \max_i d_P(x_i, y_i)$.

Proof: Each of projection maps $\Pi_i : B \longrightarrow \Delta$ is 1-Lipshitz. ∎

DEFINITION: A variety is called **Kobayashi hyperbolic** if the Kobayashi pseudometric d_K is non-degenerate.

DEFINITION: A **domain** in $\mathbb{C}^n$ is an open subset. A **bounded domain** is an open subset contained in a ball.

COROLLARY: Any bounded domain Ω in $\mathbb{C}^n$ is Kobayashi hyperbolic.

Proof: Without restricting generality, we may assume that $\Omega \subset B$ where B is an open ball. Then the Kobayashi distance in Ω is $\geqslant$ that in B. However, the Kobayashi distance in B is bounded by the metric $d(x, y) := \max_i d_P(x_i, y_i)$ as follows from above. ∎

Caratheodory metric

DEFINITION: Let $x, y \in M$ be points on a complex manifold. Define **Caratheodory pseudometric** as $d_C(x, y) = \sup\{d_P(f(x), f(y))\}$, where the supremum is taken over all holomorphic map $f : M \longrightarrow \Delta$, and d_P is Poincare metric on the disk Δ.

REMARK: Usually the term "Kobayashi/Caratheodory pseudometric" is abbreviated to "Kobayashi/Caratheodory metric", **even when it is not a metric.**

REMARK: Caratheodory pseudometric **satisfies the triangle inequality** because a supremum of pseudometrics satisfies triangle inequality.

EXERCISE: Prove that **Caratheodory pseudometric is bounded by the Kobayashi pseudometric:** $d_K \geqslant d_C$.

REMARK: Clearly, $d_C \neq 0$ on any bounded domain.

Complex hyperbolic space

DEFINITION: Let $V = \mathbb{C}^{n+1}$ be a complex vector space equipped with a Hermitian metric h of signature $(1, n)$, and $\mathbb{H}^n_{\mathbb{C}} \subset \mathbb{P}V$ projectivization of the set of positive vectors $\{x \in V \quad h(x, \bar{x}) > 0\}$. Then $\mathbb{H}^n_{\mathbb{C}}$ is equipped with a homogeneous action of $U(1, n)$. The same argument as used for space forms implies that $\mathbb{H}^n_{\mathbb{C}}$ admits a $U(1, n)$-invariant Hermitian metric, which is unique up to a constant multiplier. This Hermitian complex manifold is called **complex hyperbolic space**.

REMARK: For $n > 1$ **it is not isometric to the real hyperbolic spaces defined earlier.**

REMARK: As a complex manifold $\mathbb{H}^n_{\mathbb{C}}$ **is isomorphic to an open ball in** $\mathbb{C}^n$**.**

REMARK: The Kobayashi metric and the Caratheodory metric on $\mathbb{H}^n_{\mathbb{C}}$ are $U(1, n)$-invariant, because $U(1, n)$ acts holomorphically, hence proportional to the hyperbolic metric, which is also called **Bergman metric** on an open ball.

EXERCISE: Prove that **Kobayashi metric on a ball in** $\mathbb{C}^n$ **is equal to the Caratheodory metric.**

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Uniform convergence for Lipschitz maps

DEFINITION: A sequence of maps $f_i : \ M \longrightarrow N$ between metric spaces **uniformly converges** (or **converges uniformly on compacts**) to $f : \ M \longrightarrow N$ if for any compact $K \subset M$, we have $\lim\limits_{i \to \infty} \sup_{x \in K} d(f_i(x), f(x)) = 0$.

Claim 1: Suppose that a sequence $f_i : \ M \longrightarrow N$ of 1-Lipschitz maps converges to f pointwise in a countable dense subset $M' \subset M$. **Then f_i converges to f uniformly on compacts.**

Proof: Let $K \subset M$ be a compact set, and $N_\varepsilon \subset M'$ a finite subset such that K is a union of ε-balls centered in N_ε (such N_ε is called **an ε-net**). Then there exists N such that $\sup_{x \in N_\varepsilon} d(f_{N+i}(x), f(x)) < \varepsilon$ for all $i \geqslant 0$. Since f_i are 1-Lipschitz, this implies that

$$\sup_{y \in K} d(f_{N+i}(y), f(y)) \leqslant$$

$$\leqslant \sup_{x \in N_\varepsilon} d(f_{N+i}(x), f(x)) + \inf_{x \in N_\varepsilon} (d(f_{N+i}(x), y) + d(f(x), y)) \leqslant 3\varepsilon.$$

∎

EXERCISE: Prove that the limit f is also 1-Lipschitz.

REMARK: This proof works when M is a pseudo-metric space, as long as N is a metric space.

Arzela-Ascoli theorem for Lipschitz maps

DEFINITION: Let M, N be metric spaces. A subset $B \subset N$ is **bounded** if it is contained in a ball. A family $\{f_\alpha\}$ of functions $f_\alpha : M \longrightarrow N$ is called **uniformly bounded on compacts** if for any compact subset $K \subset M$, there is a bounded subset $C_K \subset N$ such that $f_\alpha(K) \subset C_K$ for any element f_α of the family.

THEOREM: **(Arzela-Ascoli for Lipschitz maps)**
Let $\mathcal{F} := \{f_\alpha\}$ be an infinite uniformly bounded set of 1-Lipschitz maps $f_\alpha : M \longrightarrow \mathbb{C}$, where M is a pseudo-metric space. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \longrightarrow \mathbb{C}$ uniformly.**

REMARK: The limit f is clearly also 1-Lipschitz.

Proof. Step 1: Suppose we can prove Arzela-Ascoli when M is compact. Then we can choose a sequence of compact subsets $K_i \subset M$, find subsequences in $\mathcal{F}$ converging on each K_i, and use the diagonal method to find a subsequence converging on all K_i. Therefore, **we can assume that M is bounded, and all maps $f_\alpha : M \longrightarrow \mathbb{C}$ map M into a compact subset $N \subset \mathbb{C}$.**

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Arzela-Ascoli theorem for Lipschitz maps (2)

THEOREM: (Arzela-Ascoli for Lipschitz maps)
Let $\mathcal{F} := \{f_\alpha\}$ be an infinite uniformly bounded set of 1-Lipschitz maps $f_\alpha :$ $M \longrightarrow \mathbb{C}$, where M is a pseudo-metric space. Assume that M has countable base of open sets and can be obtained as a countable union of compact subsets. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f :$ $M \longrightarrow \mathbb{C}$ uniformly.**

Proof. Step 1: We can assume that M is compact, and all maps $f_\alpha : M \longrightarrow \mathbb{C}$ **map M into a compact subset $N \subset \mathbb{C}$.**

Step 2. By definition of pointwise convergence, for any finite set $S \subset M$, there exists a subsequence f_i of $\mathcal{F}$ which converges to $f \in \mathrm{Map}(S, N)$ in S. Using diagonal method, we choose a subsequence f_i of $\mathcal{F}$ which converges to $f \in \mathrm{Map}(M', N)$ pointwise in a dense countable set $M' \subset M$. **Then f_i converges to f uniformly by Claim 1.** ∎

Normal families of holomorphic functions

DEFINITION: Let M be a complex manifold. A family $\mathcal{F} := \{f_\alpha\}$ of holomorphic functions $f_\alpha : \ M \longrightarrow \mathbb{C}$ is called **normal family** if $\mathcal{F}$ is uniformly bounded on compact subsets.

THEOREM: (Montel's theorem)
Let M be a complex manifold with countable base, and $\mathcal{F}$ a normal, infinite family of holomorphic functions. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : \ M \longrightarrow \mathbb{C}$ uniformly,** and f is holomorphic.

Proof. Step 1: As in the first step of Arzela-Ascoli, it suffices to prove Montel's theorem on a subset of M where $\mathcal{F}$ is bounded. Therefore, **we may assume that all f_α map M into a disk $\triangle$.**

Step 2: All f_α are 1-Lipschitz with respect to Kobayashi metric. Therefore, **Arzela-Ascoli theorem can be applied, giving a uniform limit $f = \lim f_i$.**

Step 3: A uniform limit of holomorphic functions is holomorphic by Cauchy formula. ∎

REMARK: The sequence $f = \lim f_i$ converges uniformly with all derivatives, again by Cauchy formula.