Riemann surfaces

lecture 7: Kobayashi metrics

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Poincaré metric on disk (reminder)

DEFINITION: Poincaré metric on a unit disk $\Delta \subset \mathbb{C}$ is an Aut(Δ)-invariant metric **(it is unique up to a constant multiplier)**.

DEFINITION: Let $f: M \longrightarrow M_1$ be a map of metric spaces. Then f is called **C-Lipschitz** if $d(x,y) \geqslant Cd(f(x), f(y))$. A map is called **Lipschitz** if it is C-Lipschitz for some $C > 0$.

THEOREM: (Schwartz-Pick lemma)
Any holomorphic map $\varphi: \Delta \longrightarrow \Delta$ from a unit disk to itself is 1-Lipschitz with respect to Poicaré metric.

Proof. Step 1: We need to prove that for each $x \in \Delta$ the norm of the differential satisfies $|D\varphi_x| \leqslant 1$. Since the automorphism group acts on Δ transitively by isometries, **it suffices to prove that $|D\varphi_x| \leqslant 1$ when $x = 0$ and $\varphi(x) = 0$.**

Step 2: This is Schwartz lemma. ■

Normal families of holomorphic functions (reminder)

DEFINITION: Let M be a complex manifold. A family $\mathcal{F} := \{f_\alpha\}$ of holomorphic functions $f_\alpha : M \longrightarrow \mathbb{C}$ is called **normal family** if $\mathcal{F}$ is uniformly bounded on compact subsets.

THEOREM: (Montel's theorem)
Let M be a complex manifold with countable base, and $\mathcal{F}$ a normal, infinite family of holomorphic functions. **Then there is a sequence $\{f_i\} \subset \mathcal{F}$ which converges to $f : M \longrightarrow \mathbb{C}$ uniformly,** and f is holomorphic.

Proof. Step 1: As in the first step of Arzela-Ascoli, it suffices to prove Montel's theorem on a subset of M where $\mathcal{F}$ is bounded. Therefore, **we may assume that all f_α map M into a disk Δ.**

Step 2: All f_α are 1-Lipschitz with respect to Kobayashi metric. Therefore, **Arzela-Ascoli theorem can be applied, giving a uniform limit $f = \lim f_i$.**

Step 3: A uniform limit of holomorphic functions is holomorphic by Cauchy formula. ■

REMARK: The sequence $f = \lim f_i$ converges uniformly with all derivatives, again by Cauchy formula.

Riemann mapping theorem

THEOREM: Let $\Omega \subset \Delta$ be a simply connected domain. **Then Ω is biholomorphic to Δ.**

Idea of a proof: We consider the Kobayashi metric on Ω and Δ, and let $\mathcal{F}$ be the set of all injective holomorphic maps $\Omega \longrightarrow \Delta$. Consider $x \in \Omega$, and let f be a map with $|df_x|$ maximal in the sense of Kobayashi metric. **Such f exists by Montel's theorem. We prove that f is a bijective isometry**, and hence biholomorphic.

PROPOSITION: Let $\mathcal{H}$ be the set of holomorphic maps $f: \Omega_1 \longrightarrow \Omega_2$ between Riemann surfaces, equipped with uniform topology, and $\mathcal{H}_0$ its subset consisting of injective maps. **Then $\mathcal{H}_0$ is closed in $\mathcal{H}$.**

Proof: Let f_i be a sequence of injective maps converging to $f: \Omega_1 \longrightarrow \Omega_2$ which is not injective. Then $f(a) = f(b)$ for some $a \neq b$ in Ω_1. Choose open disks A and B containing a and b. Then the Proposition is implied by the following lemma.

LEMMA: Let $\mathcal{R}$ be the set of all pairs of distinct holomorphic functions $f, g: \Delta \longrightarrow \mathbb{C}$ continuously extended to the boundary such that $f(x) = g(x)$ for some $x \in \Delta$. **Then $\mathcal{R}$ is open in uniform topology.**

The set of non-injective maps is open

LEMMA: Let $\mathcal{R}$ be the set of all pairs of distinct holomorphic functions $f, g: \ \Delta \longrightarrow \mathbb{C}$ continuously extended to the boundary such that $f(x) = g(x)$ for some $x \in \Delta$. **Then $\mathcal{R}$ is open in uniform topology.**

Proof. Step 1: The set of all points $z \in \Delta$ where $f = g$ is discrete **(prove it).** This implies that **we can replace Δ by a smaller disc containing x such that $f \neq g$ everywhere on its boundary.**

Step 2: Consider the function $\alpha \frac{(f-g)'}{f-g}$ on Δ. **This function has a simple pole in all the points where $f = g$.** Therefore, $n_{f,g} := \frac{1}{\pi\sqrt{-1}} \int_{\partial\Delta} \alpha dz$ is equal to the number of points $x \in \Delta$ such that $f(x) = g(x)$.

Step 3: Since the integral is continuous in unform topology, **this number is locally constant on the space of pairs such $f, g: \ \Delta \longrightarrow \mathbb{C}$.** Therefore, the set $\mathcal{R}$ of all f, g with $n_{f,g} \neq 0$ is open. ■

Coverings

DEFINITION: Constant sheaf with coefficients in a set S is a sheaf $\mathcal{F}$ such that for any $U \subset M$, the space of sections of $\mathcal{F}$ is S, independent on U. **Locally constant sheaf** is a sheaf $\mathcal{F}$ such that each point $x \in M$ has a neighbourhood $U \subset M$ such that the restriction of $\mathcal{F}$ to U is constant.

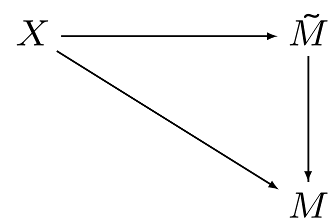
DEFINITION: A continuous map $\pi: \tilde{M} \longrightarrow M$ of topological spaces is a **covering** if π is locally a diffeomorphism, and the space of sections of π is a locally constant sheaf.

EXAMPLE: The map $x \longrightarrow x^2$ is a covering from $\mathbb{C}^* := \mathbb{C}\backslash 0$ to itself **(prove it)**.

Homotopy lifting principle

DEFINITION: A topological space X is **locally path connected** if for each $x \in X$ and each neighbourhood $U \ni x$, there exists a smaller neighbourhood $W \ni x$ which is path connected.

THEOREM: (homotopy lifting principle)
Let X be a simply connected, locally path connected topological space, and $\tilde{M} \longrightarrow M$ a covering map. Then for each continuous map $X \longrightarrow M$, there exists a lifting $X \longrightarrow \tilde{M}$ making the following diagram commutative.



COROLLARY: Let $\varphi : \Omega \longrightarrow \mathbb{C}^*$ be a holomorphic map from a simply connected domain Ω. **Then there exists a holomorphic map $\varphi_1 : \Omega \longrightarrow \mathbb{C}^*$ such that for all $z \in \Delta$, $\varphi(z) = \varphi_1(z)^2$.**

Proof: We apply homotopy lifting principle to $X = \Omega$, $M = \tilde{M} = \mathbb{C}^*$, and $\tilde{M} \longrightarrow M$ mapping x to x^2. ■

REMARK: We denote $\varphi_1(z)$ by $\sqrt{\varphi(z)}$, for obvious reasons.

Kobayashi metric and the map $x \longrightarrow x^2$

CLAIM: Consider a non-bijective injective holomorphic map $\varphi: \Delta \longrightarrow \Delta$ from Poincare disk to itself. **Then $|d\varphi| < 1$ at each point,** where $d\varphi$ is a norm of an operator $d\varphi: T_x\Delta \longrightarrow T_{\varphi(x)}\Delta$ taken with respect to the Poincare metric.

Proof: Let $\varphi: \Delta \longrightarrow \Delta$ be a holomorphic map which satisfies $|d\varphi| = 1$ at $x \in \Delta$. Replacing φ by $\gamma_1 \circ \varphi \circ \gamma_2$ if necessary, where γ_i are biholomorphic isometries of Δ, we may assume that $x = 0$ and $\varphi(x) = 0$. By Schwartz lemma for such φ, $|d\varphi(0)| = 1$ implies that φ is a linear biholomorphic map. ■

Corollary 1: Let $\varphi: \Delta \longrightarrow \Delta \backslash 0$ be a holomorphic function, and $\sqrt{\varphi}$ a holomorphic function defined above. Let $|d\varphi|(x)$ denote the norm of the operator $d\varphi$ at $x \in \Delta$ computed with respect to the Poincare metric on Δ. **Then $|d\varphi|(x) < |d\sqrt{\varphi}|(x)$ for any $x \in \Delta$.**

Proof: Let $\psi(x) = x^2$. By the claim above, $|d\psi|(x) < 1$ for all $x \in \Delta$. Using the chain rule, we obtain that $d\varphi = d\psi \circ d\sqrt{\varphi}$. which gives $|d\varphi|(x) = |d\psi|(\sqrt{\varphi}(x))|d\sqrt{\varphi}|(x)$, hence

$$|d\sqrt{\varphi}|(x) = \frac{|d\varphi|(x)|}{|d\psi|(\sqrt{\varphi}(x))} > |d\varphi|(x).$$

■

Riemann mapping theorem

THEOREM: Let $\Omega \subset \Delta$ be a simply connected domain. **Then Ω is biholomorphic to Δ.**

Proof. Step 1: Consider the Kobayashi metric on Ω and Δ, and let $\mathcal{F}$ be the set of all injective holomorphic maps $\Omega \longrightarrow \Delta$. Consider $x \in \Omega$, and let f be a map with $|df|(x)$ maximal in the sense of Kobayashi metric. **Such f exists by Montel's theorem.** Since f lies in the closure of $\mathcal{F}$, and the set of injective maps is closed, **f is injective.**

Step 2: It remains to show that f is surjective. Suppose it is not surjective: $z \notin f(\Omega)$. Taking a composition of f and an isometry of the Poincare disk does not affect $|df|(x)$, hence we may assume that $z = 0$. **Then the function $\sqrt{f}$ is a well defined holomorphic map from Ω to Δ.** By Corollary 1, $|d\sqrt{f}|(x) > |df|(x)$, which is impossible, because it $|df|(x)$ is maximal. ■

Fatou and Julia sets

DEFINITION: Let $f: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a rational map, and $\{f^i\} = \{f, f \circ f, f \circ f \circ f, ...\}$ the set of all iterations of f. **Fatou set of f** is the set of all points $x \in \mathbb{C}P^1$ such that for some neighbourhood $U \ni x$, the restriction $\{f^i|_U\}$ is a normal family, and **Julia set** is a complement to Fatou set.

EXAMPLE: For the map $f(x) = x^2$, Julia set is the unit circle, and the Fatou set is its complement **(prove it)**.

DEFINITION: **Attractor point** z is a fixed point of f such that $|df|(z) < 1$; the **attractor basin** for z is the set of all $x \in \mathbb{C}P^1$ such that $\lim_i f^i(x) = z$.

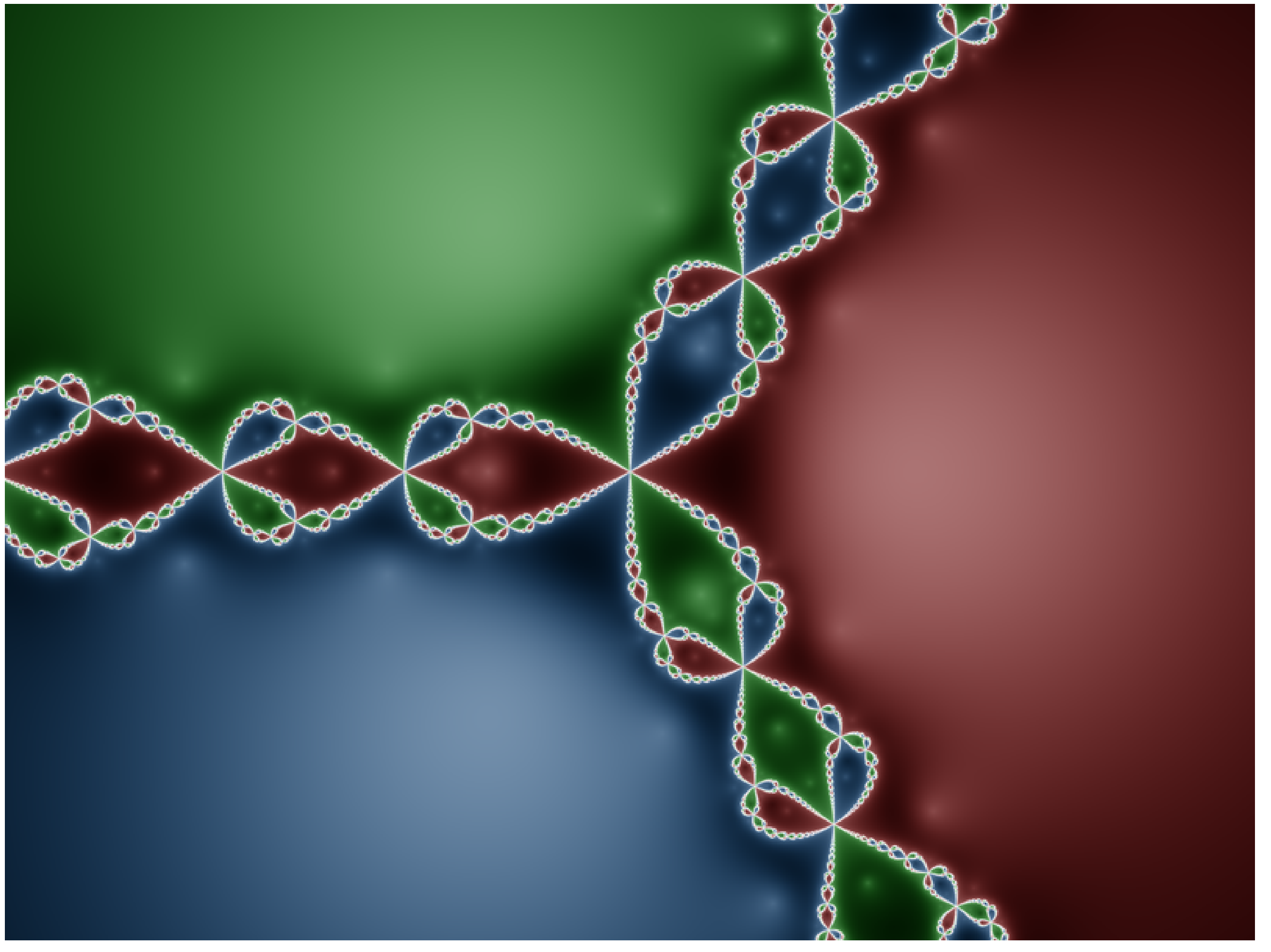
CLAIM: For any fixed point z, **its attractor basin belongs to the Fatou set.**

Proof: Indeed, since $\lim_i f^i(x) = z$ for any point in attractor basin U, $\{f^i\}$ is a normal family on U **(pointwise convergence is equivalent to uniform convergence for bounded holomorphic functions by Arzela-Ascoli theorem).** ■

DEFINITION: **Newton iteration** for solving the polynomial equation $g(z) = 0$: a solution is obtained as a limit $\lim_i f^i(z)$, where $f(z) = z - \frac{g(z)}{g'(z)}$. Indeed, solutions of $g(z) = 0$ are attracting fixed points of f **(check this)**.

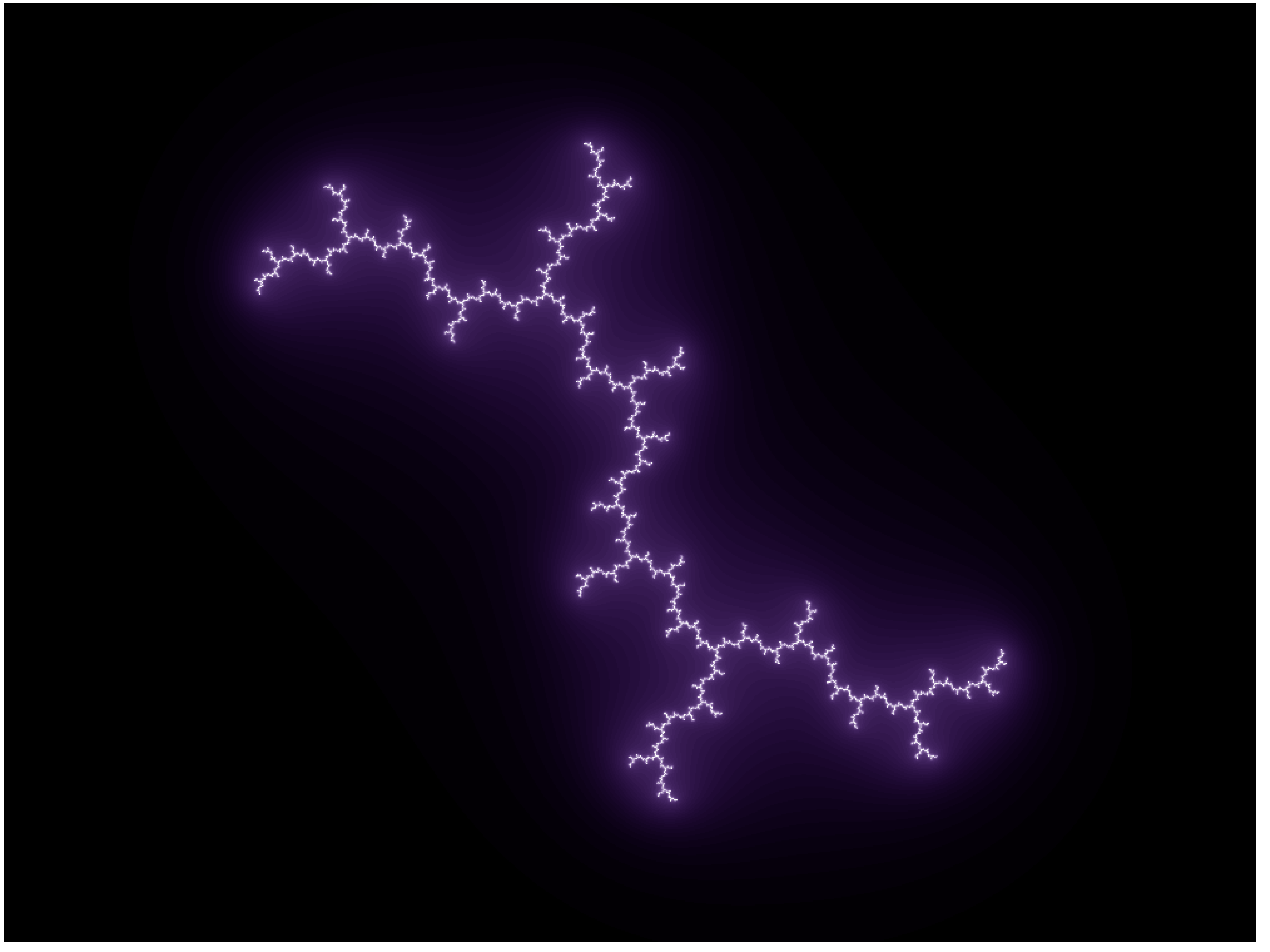
Fatou and Julia sets for $f(z) = \frac{1+2z^3}{3z^2}$

We apply the Newton iteration method to $g(z) = z^3 - 1$.



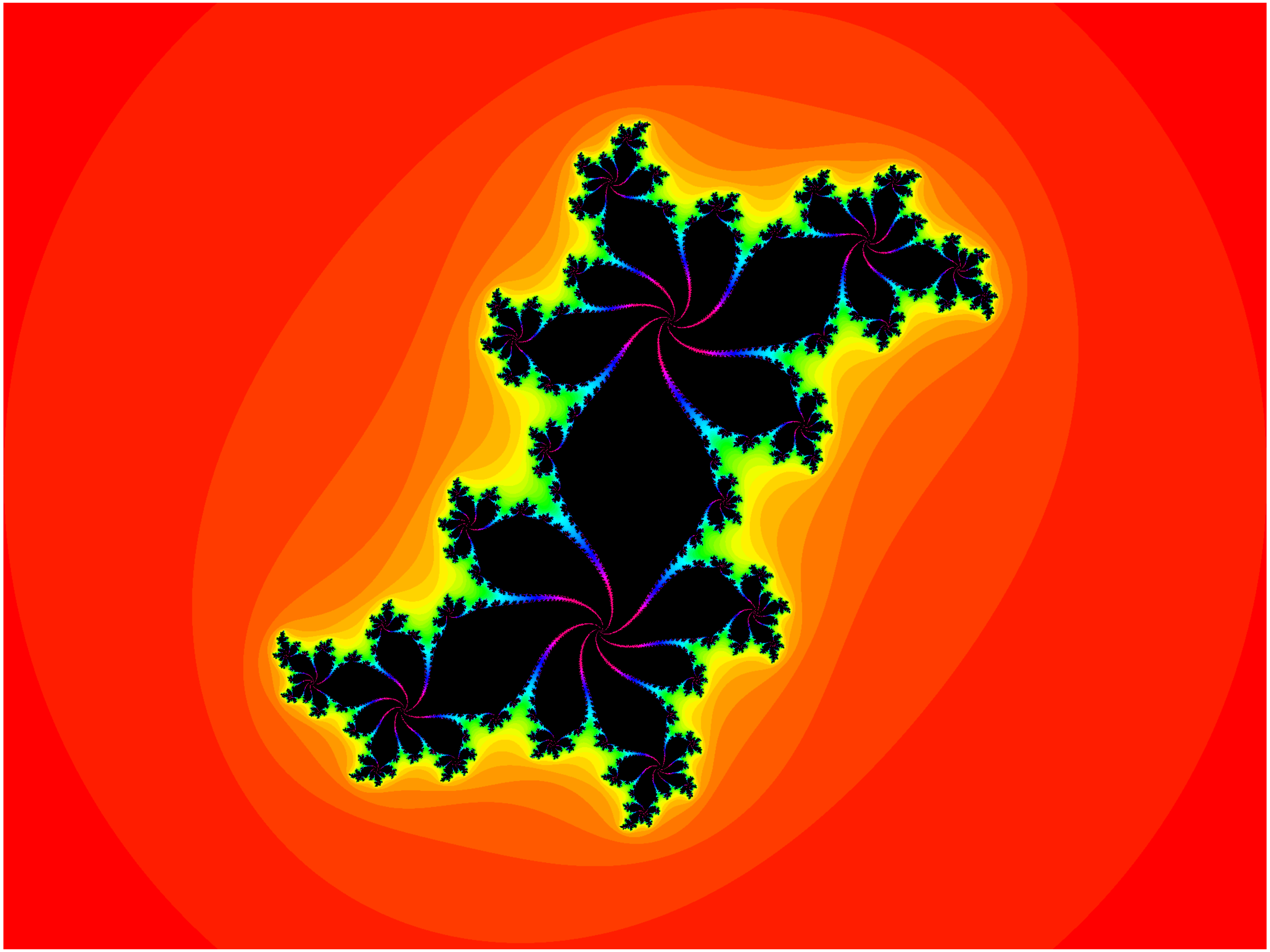
Julia set (in white) for the map $f(z) = \frac{1+2z^3}{3z^2} = z - \frac{g(z)}{g'(z)}$. Coloring of Fatou set according to attractor (the roots of $g(z) = z^3 - 1$).

Julia set for $f(z) = z^2 - \sqrt{-1}$

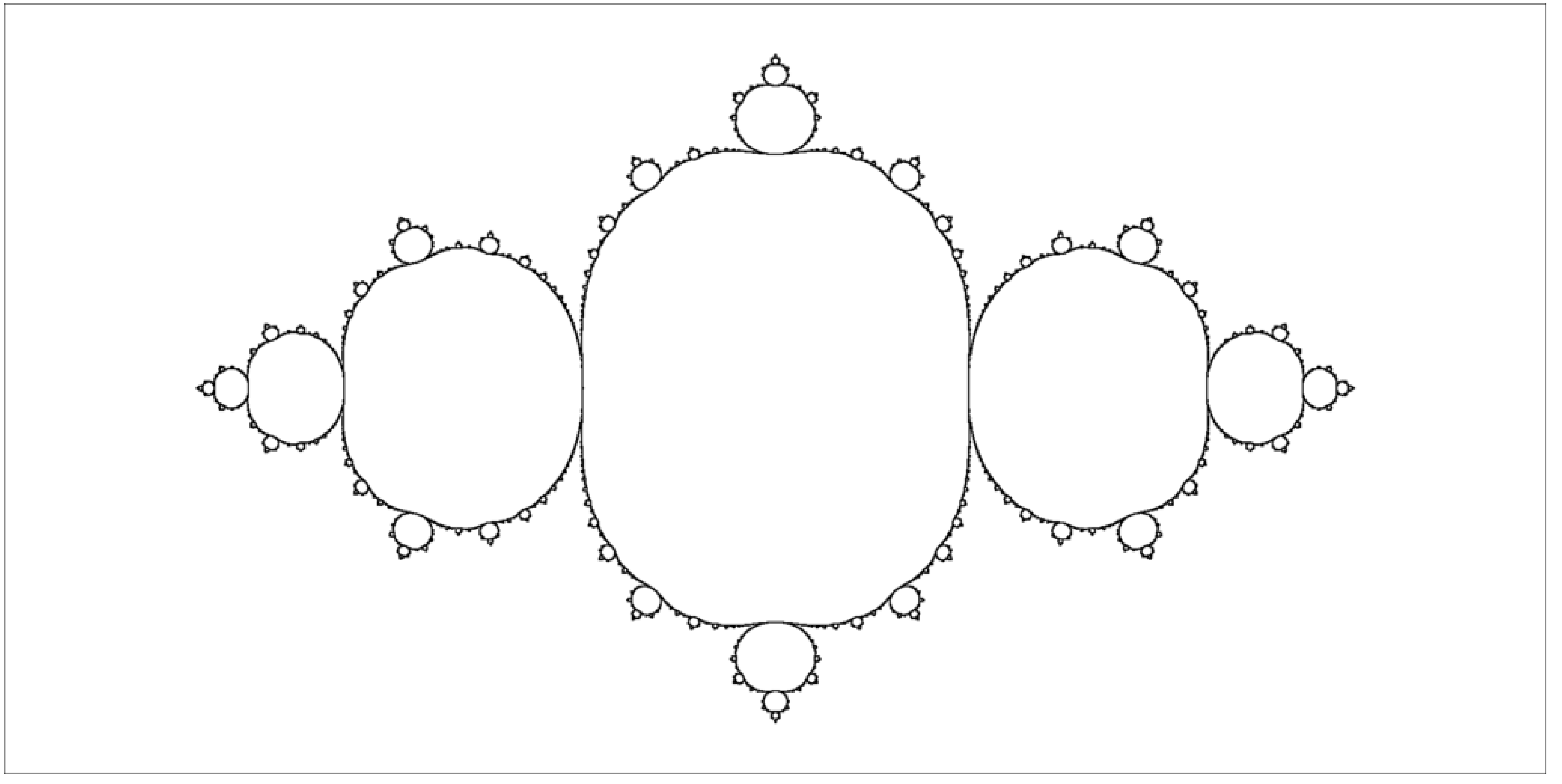


Julia set for $f(z) = z^2 - \sqrt{-1}$ is called **dendrite**.

Julia set for $f(z) = z^2 + 0.12 + 0.6\sqrt{-1}$



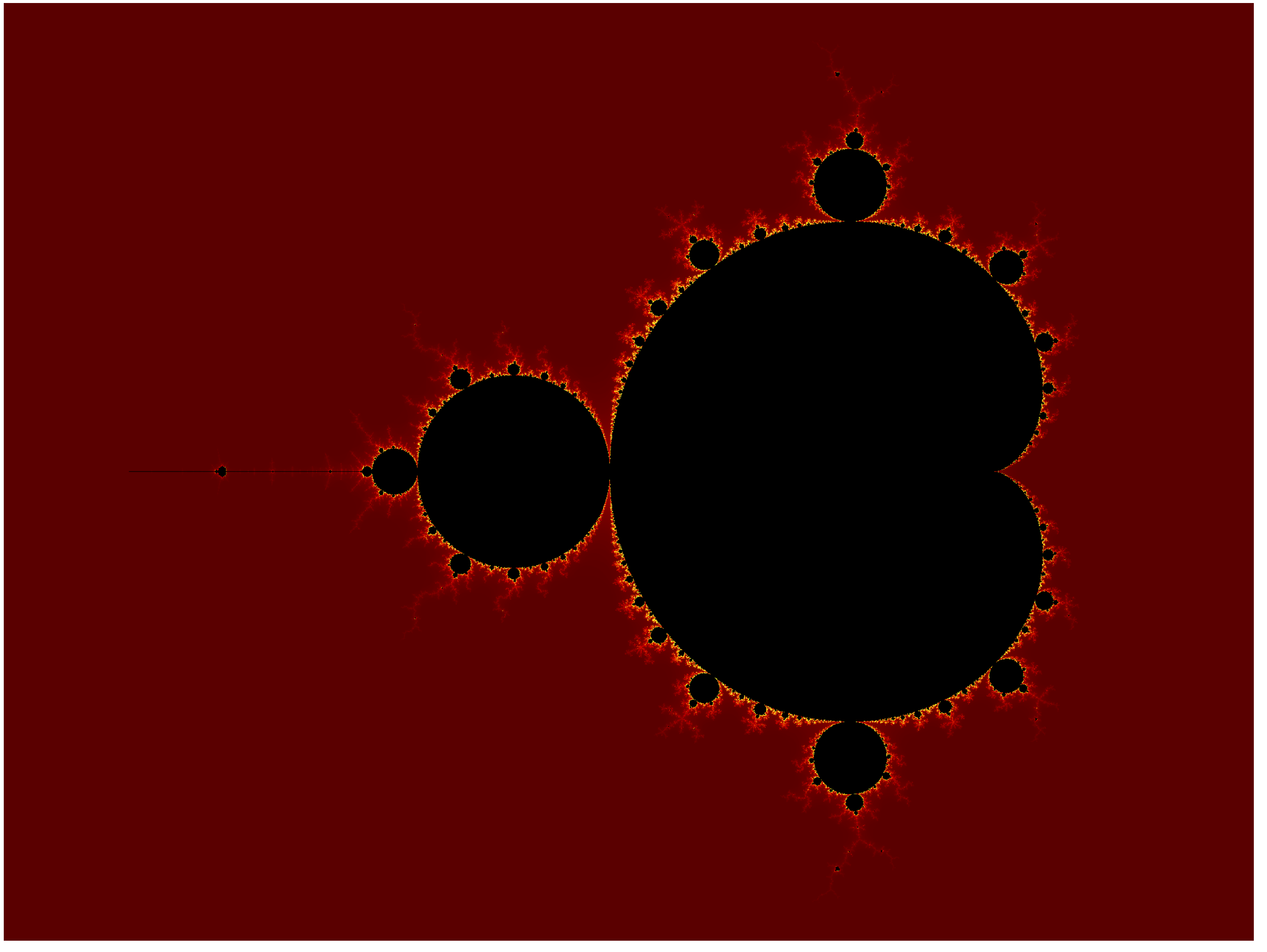
San Marco fractal



San Marco fractal is the Julia set for $f(z) = z^2 - 0.75$

Mandelbrot set

DEFINITION: **Mandelbrot set** is the set of all c such that 0 belongs to the Fatou set of $f(z) = z^2 + c$.



Properties of Fatou and Julia sets

REMARK: Let $f : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a holomorphic map. **Then the Fatou $F(f)$ and Julia set $J(f)$ of f are f-invariant.**

LEMMA: (Iteration lemma) For each k, $J(f) = J(f^k)$, where f^k is k-th iteration of f.

Proof. Step 1: Clearly, $F(f^k) \subset F(f)$, because $\overline{\{f^k, f^{2k}, f^{3k}, ...\}}$ is compact when $\overline{\{f, f^2, f^3, ...\}}$ is compact.

Proof: Conversely, suppose that $X = F(f^k)$; then $\overline{\{f^k, f^{2k}, f^{3k}, ...\}}$ is compact, but then $\overline{\{f, f^{k+1}, f^{2k+1}, f^{3k+1}, ...\}}$ is also compact as a continuous image image of a compact (the composition is continuous in uniform topology), same for $\overline{\{f^2, f^{k+2}, f^{2k+2}, f^{3k+2}, ...\}}$, and so on. Then $\overline{\{f, f^2, f^3, ...\}}$ is obtained as a union of k compact sets. **Therefore, $F(f) \subset F(f^k)$.** ■

Properties of Fatou and Julia sets (2)

THEOREM: Julia set of polynomial map $f: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ is non-empty, unless $\deg f \leqslant 1$.

Proof: Let $\Delta \subset \mathbb{C}P^1$, and $n(g)$ the number of critical points of a holomorphi function g in Δ. Then $n(g) = \frac{1}{\pi\sqrt{-1}} \int_{\partial\Delta} \frac{g'}{g} dz$, and this number is locally constant in uniform topology if g has no critical points on the boundary. Since the number of critical points of f^i is $i \deg f - 1$, it converges to infinity, hence f^i **cannot converge to a holomorphic function everywhere.** ■